Return to Equilibrium for Pauli-Fierz Systems

Jan Dereziński[1] and Vojkan Jakšić[2]

[1]Department of Mathematical Methods in Physics
Warsaw University, Hoża 74
00-682 Warszawa, Poland

[2]Department of Mathematics and Statistics
McGill University, 805 Sherbrooke Street West
Montreal, QC, H3A 2K6, Canada

May 27, 2003

Abstract

We study ergodic properties of Pauli-Fierz systems—W^*-dynamical systems often used to describe the interaction of a small quantum system with a bosonic free field at temperature $T \geq 0$. We prove that, for a small coupling constant uniform as the positive temperature $T \downarrow 0$, a large class of Pauli-Fierz systems has the property of return to equilibrium. Most of our arguments are general and deal with mathematical theory of Pauli-Fierz systems for an arbitrary density of bosonic field.

Acknowledgments. The research of the first author was partly supported by the Postdoctoral Training Program HPRN-CT-2002-0277 and the grant SPUB127. A part of this work was done during a visit of the first author to Aarhus University supported by MaPhySto funded by the Danish National Research Foundation, during his visit to University of Montreal and during a visit of both authors to Schrödinger Institute in Vienna and Johns Hopkins University in Baltimore. The research of the second author was partly supported by NSERC.

1 Introduction

A quantum system is often described by a W^*-algebra $\mathfrak{M}$ with a σ-weakly continuous group of automorphisms $t \mapsto \tau^t$. The pair $(\mathfrak{M}, \tau)$ is called a W^*-dynamical system and τ a W^*-dynamics. We say that the system $(\mathfrak{M}, \tau)$ has the property of return to equilibrium if there exists a normal state ω on $\mathfrak{M}$ such that for all normal states ϕ and $A \in \mathfrak{M}$,

$$\lim_{|t|\to\infty} \phi(\tau^t(A)) = \omega(A).$$

Such ω is obviously unique and τ-invariant. Physical intuition suggests the following quasitheorem.

Quasitheorem *Suppose that* $(\mathfrak{M}, \tau)$ *describes a quantum system that is:*

(1) *infinitely extended;*

(2) *a localized perturbation of a thermal equilibrium system;*

(3) *sufficiently regular;*

(4) *sufficiently generic.*

Then $(\mathfrak{M}, \tau)$ *has the property of return to equilibrium.*

Conditions (1) and (3) are idealizations necessary to prove sharp mathematical results. In particular, it is well-known that finite volume (confined) quantum systems do not return to equilibrium.

Condition (2) is related to the issue of stability of equilibrium states (see [BR2] and references therein). It is expected on physical grounds, and in some circumstances it can been proven, that if $(\mathfrak{M}, \tau)$ describes a localized perturbation of a physical system away from thermal equilibrium, then there are no normal τ-invariant states (see Subsections 3.6 and 7.9).

Concerning (4), some assumptions are necessary to prevent the existence of internal symmetries which would lead to an artificial multiplicity of τ-invariant normal states. In our paper the conditions of this type will be called *effective coupling conditions* and they will be generically satisfied.

In this paper we will study a class of quantum systems which are commonly used to describe the interaction of a "small" quantum system, often called an "atom", with a "bosonic reservoir". We will call them Pauli-Fierz systems [PF]. They arise in physics as simplified versions of the non-relativistic QED.

We note that in the literature the name "Pauli-Fierz Hamiltonians" appears in a number of different (although closely related) contexts. Our definition of Pauli-Fierz systems is consistent with our previous work [DG, DJ1].

Our main result is a precise formulation of the conditions decribed in the "quasitheorem" and a proof that under these conditions Pauli-Fierz systems have the property of return to equilibrium. Results closely related to ours can be found in [BFS2, JP2, M] and we will discuss them in Subsection 1.2. The rest of this section is devoted to an informal discussion of our main results.

In our paper the small system is described by a finite dimensional Hilbert space $\mathcal{K}$ and a Hamiltonian K.

The bosonic reservoir is described by a pair $(\mathcal{Z}, h)$ where $\mathcal{Z}$ and h are the Hilbert space and the energy operator of a single boson. We will always assume that $h \geq 0$. Physically, the bosons can be interpreted as phonons or photons.

The interaction between the small system and the reservoir is specified by a form-factor λv, where $v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{Z})$ and λ is a real coupling constant which controls the strength of the interaction. Our main results hold for sufficiently small nonzero values of λ.

The data $(\mathcal{K}, K, \mathcal{Z}, h, v)$ determine the basic Pauli-Fierz Hamiltonian, which is defined as the self-adjoint operator

$$H = K \otimes 1 + 1 \otimes \mathrm{d}\Gamma(h) + \lambda V$$

on the Hilbert space $\mathcal{H} = \mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z})$, where $\Gamma_{\mathrm{s}}(\mathcal{Z})$ is the bosonic Fock space over the 1-particle space $\mathcal{Z}$ and the interaction term V is the field operator associated to the form-factor v. Thus we obtain the W^*-dynamical system

$$\left(\mathcal{B}(\mathcal{H}),\ \mathrm{e}^{\mathrm{i}tH} \cdot \mathrm{e}^{-\mathrm{i}tH}\right). \tag{1.1}$$

The W^*-dynamical system (1.1) is however not our main object of study. We are interested in a family of W^*-dynamical systems that arise as thermodynamical limits of (1.1) and which describe Pauli-Fierz systems with non-zero radiation field density. Apart from $(\mathcal{K}, K, \mathcal{Z}, h, v)$, these systems are parametrized by a positive operator (the radiation density operator) ρ on $\mathcal{Z}$ commuting with h. We call them Pauli-Fierz systems at density ρ. To describe such systems one needs to use the so-called Araki-Woods representations of CCR [AW, BR2]. In typical cases, for instance if ρ has some continuous spectrum, the corresponding W^*-algebras are of type III. The system (1.1) corresponds to the density $\rho = 0$ and its W^*-algebra $\mathcal{B}(\mathcal{H})$ is of type I.

A special class of radiation densities is given by Planck's law at inverse temperature β, that is $\rho_\beta = (e^{\beta h} - 1)^{-1}$. Such densities describe a system close to thermal equilibrium at temperature $1/\beta$. A large part of our paper is not restricted to the thermal case and deals with an arbitrary radiation density. These results are useful in the study of non-equilibrium theory of Pauli-Fierz systems.

For shortness, in the remaining part of the introduction we restrict ourselves to the case of thermal densities. The main object of our study is a 1-parameter family of W^*-dynamical systems

$$(\mathfrak{M}_\beta, \tau_\beta) \tag{1.2}$$

where $\mathfrak{M}_\beta = \mathcal{B}(\mathcal{K}) \otimes \mathfrak{M}_{\beta,1}^{\mathrm{AW}}$, and $\mathfrak{M}_{\beta,1}^{\mathrm{AW}}$ is the Araki-Woods W^*-algebra corresponding to the density ρ_β. The dynamics is defined in a canonical way given the data $(\mathcal{K}, K, \mathcal{Z}, h, v)$ and the inverse temperature $\beta \in]0, \infty]$. For $\beta = \infty$ the system (1.2) coincides with (1.1). Under the conditions used in our paper the W^*-dynamical systems (1.2) are non-equivalent for distinct β.

The Pauli-Fierz systems (1.2) considered in our paper satisfy the first two conditions of the "quasitheorem". They describe an infinitely extended system (this is expressed in particular by the fact that h has continuous spectrum). Since the radiation density of the bosonic field is given by the Planck law, the system is near thermal equilibrium.

The information on the W^*-dynamics τ_β is conveniently encoded by a certain self-adjoint operator L_β called the Liouvillean. The operator L_β is canonically defined in the standard representation of $\mathfrak{M}_\beta$.

For $\beta < \infty$ (positive temperatures), under quite broad conditions one can show the existence of a (τ_β, β)-KMS vector, which is an eigenvector of L_β with a zero eigenvalue. This result was proven in [DJP] and is based on an extension of the well-known result of Araki [Ar, BR2].

For $\beta = \infty$ (the zero temperature) in many cicumstances one can show that the Pauli-Fierz Hamiltonian H has a ground state [AH, BFS1, Ge, Sp2, Sp3]. The ground state gives rise to an eigenvector of the corresponding Liouvillean L_∞ with a zero eigenvalue.

For $\beta < \infty$, the return to equilibrium can be deduced from spectral properties of L_β. In particular, the return to equilibrium follows if L_β has no singular spectrum except for a nondegenerate eigenvalue at zero.

For a W^*-dynamical system $(\mathfrak{M}, \tau)$ with $\mathfrak{M}$ being a type I factor, the return to equilibrium never holds (unless the algebra is 1-dimensional). Therefore, the Pauli-Fierz systems with $\beta = \infty$ do not have the return to equilibrium property.

In the literature [HS1, FGS], one can find a related property called the relaxation to a ground state, which in some cases can be proven for zero temperature systems. Note, however, that to prove this property one needs to consider appropriate C^*-dynamical systems, whereas we always consider W^*-dynamical systems.

The regularity assumption on Pauli-Fierz systems that we make is based on the ideas and results of [JP1, JP2]. This method consists in "gluing" together negative and positive frequences of the bosons, which allows to define a "conjugate operator"—the generator of translations in the spectral variables. In the original approach of [JP1, JP2] the analyticity of the form-factor with respect to the translations was assumed and the Liouvillean was studied using the complex deformation method. Here, we asssume only the differentiability of a sufficiently high order and study the spectrum using the Mourre theory developed for this purpose in our previous paper [DJ1]. The Mourre theory allows us to treat Pauli-Fierz systems more efficiently, especially at low temperatures.

To express our effective coupling assumptions we use some simple algebraic conditions derived from the so-called Fermi Golden Rule, which describes how to compute eigenvalues and resonances to the second order. In particular, it can be used to predict which eigenvalues will disappear after the interaction is "switched on". The information obtained by the Fermi Golden Rule can be conveniently encoded in the so-called Level Shift Operator Γ_β—an object that plays a crucial role in our paper.

The Liouvillean of a Pauli-Fierz system in the absence of interaction has a large kernel (of dimension at least $\dim \mathcal{K}$). After the interaction is "switched on", the dimension of the kernel of L_β is at least one. Our aim is to show that there are no other eigenvectors of L_β for small nonzero λ.

For small nonzero λ and all $\beta \in]0, \infty[$, that is for the whole range of positive temperatures, analysis of the Level Shift Operator Γ_β gives a single condition that on the formal level indicates the absence of the singular spectrum of L_β except for a nondegenerate eigenvalue at zero. In order to check this condition one constructs a certain (finite dimensional) $*$-algebra $\mathfrak{N} \subset \mathcal{B}(\mathcal{K})$ which depends only on

$(\mathcal{K}, K, \mathcal{Z}, h, v)$ and not on the inverse temperature β. Our positive temperature effective coupling assumption is that $\mathfrak{N} = \mathbb{C}1$. The result of [DJ1] provides a rigorous method to show that the above assumption together with a sufficient regularity of the form-factor imply the desired spectral properties of L_β, and hence imply the return to equilibrium. The result described so far is, however, not uniform in the temperature.

Our main goal is to show that under suitable conditions Pauli-Fierz systems have the property of return to equilibrium uniformly in the temperature. This requires a detailed analysis of the zero temperature case, which is in many respects different from that of positive temperatures. Analysis of the Level Shift Operator Γ_∞ yields natural effective coupling conditions under which one should expect that for a sufficiently small nonzero λ the Pauli-Fierz Hamiltonian H has no singular spectrum except for a nondegenerate ground state. These conditions involve the nondegeneracy of the unperturbed ground state and the strict positivity of a certain auxiliary operator. The result of [DJ1] gives a rigorous proof that these conditions together with a sufficient regularity of the form-factor imply the desired properties of Pauli-Fierz Hamiltonians. As an immediate consequence, under the same conditions zero temperature Liouvilleans have no singular spectrum except possibly for a nondegenerate eigenvalue at zero.

If the zero temperature *and* the positive temperature effective coupling assumptions hold and if the form-factor is sufficiently regular, then we can establish return to equilibrium uniformly in the temperature. More precisely, we show that for any $\beta_0 > 0$ there is $\lambda_0 > 0$ such that for $0 < |\lambda| < \lambda_0$ and $\beta \in [\beta_0, \infty[$ the Pauli-Fierz Liouvillean L_β has no singular spectrum except for a nondegenerate zero eigenvalue. It follows that under the same conditions the system $(\mathfrak{M}_\beta, \tau_\beta)$ has the property of return to equilibrium and this is the main result of our paper.

We emphasize that the above mentioned effective coupling conditions are important ingredients of our approach. These conditions are optimal if one considers only the 2nd order perturbation theory. They are quite simple algebraically and it is perhaps surprising that a single effective coupling condition applies to all positive temperatures.

Similar results can be given if we consider radiation densities that are not necessarily given by the Planck law. For instance, we show that if the small system interacts with two bosonic reservoir at distinct temperatures, then generically the coupled system has no normal time-invariant states.

1.1 Organization of the paper

In Section 2 we briefly review the definitions and results of the theory of W^*-algebras needed in our paper. In particular, we quote the results of [DJP].

In Section 3 we give a simplified presentation of our main results. To make the paper more accessible, in this section we restrict ourselves to the case of a scalar massless field. This section is not used in the remaining part of the paper, where a more general class of models is considered and a different notation is used. Section

3 serves as a quick introduction to our results and allows to compare them easily with the results existing in the literature.

In Section 4 we introduce the notation and review some basic facts which we will need in the paper. In Section 5 we introduce Pauli-Fierz operators and review their properties following [DJ1].

In Section 6 we introduce Pauli-Fierz systems at density ρ, compute Pauli-Fierz Liouvilleans L_ρ and study their properties. The main technical results of this section concern the structure of the Level Shift Operator Γ_ρ of the Liouvillean L_ρ.

In Section 7 we discuss thermal Pauli-Fierz systems. In Subsection 7.4 we give conditions under which thermal Pauli-Fierz systems have the property of return to equilibrium for a fixed inverse temperature. The result uniform in the temperature is described in Subsection 7.7.

As we have already mentioned, the main regularity assumption our method requires concerns the gluing condition of [JP1, JP2, DJ1]. In Section 8 we discuss the gluing condition in the context of scalar and vector massless bosons.

1.2 Comparison with the literature

Hamiltonians similar to those considered in our paper appear frequently in the physics literature, see e.g.[PF]. In the recent years there has been a revival of interest in rigorous results about these operators, starting with such papers as [BFS1, DG, HS2, JP1, Sk].

From the technical point of view, the results of our paper concern mainly spectral properties of a certain class of Pauli-Fierz operators. A large part of the literature on spectral analysis of Pauli-Fierz operators can be divided into two classes. The first uses the generator of translations as the main tool and the second uses the generator of dilations.

In the context of Pauli-Fierz systems, the generator of translations was used for the first time in [JP1, JP2], where it was applied to deform analytically positive temperature Pauli-Fierz Liouvilleans. These papers also contain the first proof that thermal Pauli-Fierz systems have the property of return to equilibrium. The infinitesimal version of this method based on the Mourre theory was developed in [DJ1]. That paper was a technical preparation for the present paper. In fact, in the introduction to [DJ1] we roughly described the applications contained in this paper (without, however, giving exact conditions).

The generator of translations is also the main tool of an interesting paper by Merkli [M], which is devoted to the proof of return to equilibrium in the mean. This paper is based on the technique of a “modified conjugate operator” originally due to Hübner and Spohn [HS2] and elaborated later in [BFSS]. The results of [M] are closely related to ours. One of the differences is that Merkli studies return to equilibrium in the mean and he does not show the absence of singular continuous spectrum for Pauli-Fierz Liouvilleans. His proof is based on the virial theorem, whereas the method of [DJ1] is based on the limiting absorbtion principle. Merkli’s main result is not uniform in the temperature.

In the context of Pauli-Fierz systems with positive mass the generator of dilations was used for the first time in [OY]. In the zero temperature massless case it was used first in [BFS1]. In [BFS2] the generator of dilations was used to study return to equilibrium of Pauli-Fierz systems uniformly in the temperature. A distinctive feature of the papers [BFS1, BFS2] is the so-called renormalization group technique, which in this context is meant to describe an iterative procedure based on the Feshbach method, used to control the spectrum of Pauli-Fierz operators. The results of [BFS2] resemble closely ours and Merkli's. Strictly speaking, however, the conditions of [BFS2] are not comparable to ours and one can find interactions which can be treated with one method and not by the other. With regard to the infrared singularity, the conditions of [BFS2] for the uniform in temperature return to equilibrium are somewhat less restrictive than ours.

There is a vast body of literature dealing with Pauli-Fierz systems in the Van Hove weak coupling limit $s = \lambda^2 t$, $\lambda \downarrow 0$, with s fixed (see eg. [Da, Sp2]). In this limit one obtains an irreversible Markovian dynamics on the algebra of the small system. The generator of this dynamics is sometimes called the Davies generator. The Level Shift Operator, which arises through the Fermi Golden Rule for the Liouvillean and is one of the main tools of our paper, is similar to the Davies generator in many respects. Let us stress, however, that they are different operators. The Davies generator describes the evolution of observables and always has a zero eigenvalue. On the other hand, the Level Shift Operator describes the shift of eigenvalues and resonances of the Liouvillean and often does not have a zero eigenvalue. In the thermal case, however, these operators are closely related, see [DJ2].

The effective coupling assumptions for return to equilibrium used in our paper are different from those found in the literature. To our knowledge they are simpler and less restrictive—in fact, they are optimal in the context of the 2nd order perturbation theory. They are based on a detailed algebraic analysis of the Level Shift Operator for Pauli-Fierz Liouvilleans that seems to appear for the first time in the literature. Somewhat similar effective coupling conditions for return to equilibrium of quantum Markovian semigroups were given in [Fr, Sp2].

One of the consequences of our method is a relatively simple proof of the nondegeneracy of the ground state of Pauli-Fierz Hamiltonians under certain regularity and effective coupling assumptions. The other proofs in the literature use Perron-Frobenius type arguments and are restricted to positivity preserving interactions [BFS1, Sp1, Sp3]. The only exception that we know is the proof based on the "renormalization group" contained in [BFS1].

The first result about existence of KMS states for Pauli-Fierz systems goes back to [FNV] where the spin-boson system was considered. It was also proven in [BFS2] under a more restrictive infrared condition than that of our paper.

Our result about a system coupled to several reservoirs at different temperatures can be compared with recent works on non-equilibrium quantum statistical physics [JP3, Ru]. Note that these papers use C^*-dynamical systems rather than W^*-dynamical systems and look for stationary states that are not normal.

2 Algebraic preliminaries

In this section we review some elements of the theory of W^*-algebras needed in our paper. For more details we refer the reader to [DJP], and also [BR1, BR2, St].

One of the most important concepts of the modern theory of W^*-algebras is the so-called standard representation. We say that a quadruple $(\pi, \mathcal{H}, J, \mathcal{H}_+)$ is a standard representation of a W^*-algebra $\mathfrak{M}$ if $\pi : \mathfrak{M} \to \mathcal{B}(\mathcal{H})$ is a $*$-representation, J is an antiunitary involution on $\mathcal{H}$ and $\mathcal{H}_+$ is a self-dual cone in $\mathcal{H}$ satisfying the following conditions:
(1) $J\pi(\mathfrak{M})J = \pi(\mathfrak{M})'$;
(2) $J\pi(A)J = \pi(A)^*$ for A in the center of $\mathfrak{M}$;
(3) $J\Psi = \Psi$ for $\Psi \in \mathcal{H}_+$;
(4) $\pi(A)J\pi(A)\mathcal{H}_+ \subset \mathcal{H}_+$ for $A \in \mathfrak{M}$.
Every W^*-algebra has a unique (up to unitary equivalence) standard representation.

The standard representation has several important properties. First, every normal state ω has a unique vector representative in $\mathcal{H}_+$ (there is a unique normalized vector $\Omega \in \mathcal{H}_+$ such that $\omega(A) = (\Omega|\pi(A)\Omega)$). Secondly, for every W^*-dynamics τ on $\mathfrak{M}$ there is a unique self-adjoint operator L on $\mathcal{H}$ such that

$$\pi(\tau^t(A)) = \mathrm{e}^{\mathrm{i}tL}\pi(A)\mathrm{e}^{-\mathrm{i}tL}, \qquad \mathrm{e}^{\mathrm{i}tL}\mathcal{H}_+ = \mathcal{H}_+. \tag{2.3}$$

The operator L is called the Liouvillean of the W^*-dynamical system $(\mathfrak{M}, \tau)$.

Theorem 2.1 *Let ω be a normal state and $\Omega \in \mathcal{H}_+$ its vector representative. Then ω is τ-invariant iff $\Omega \in \mathrm{Ker}L$.*

Theorem 2.2 (1) *The Liouvillean L has no eigenvalues iff the W^*-dynamics τ has no normal invariant states.*

(2) *The Liouvillean L has exactly one nondegenerate eigenvalue at zero iff the W^*-dynamics τ has a single normal invariant state.*

(3) *Suppose L has no singular spectrum except for a nondegenerate eigenvalue at zero, and that the corresponding eigenstate is separating for $\mathfrak{M}$. Then the system $(\mathfrak{M}, \tau)$ has the property of return to equilibrium.*

Theorem 2.1 follows easily from (2.3). Theorem 2.2 (3) was proven in [JP1] although similar results can be traced to much older literature (see [BR1, Ja]).

We now describe some results concerning perturbation theory of W^*-dynamical systems. Our presentation follows [DJP].

Let $(\mathfrak{M}, \tau)$ be a W^*-dynamical system and $(\pi, \mathcal{H}, J, \mathcal{H}_+)$ a standard representation of $\mathfrak{M}$. Let L be the Liouvillean of τ. Let V be a self-adjoint operator affiliated to $\mathfrak{M}$. Let us state the following assumption:

Assumption 2.A $L + \pi(V)$ *is essentially self-adjoint on* $\mathcal{D}(L) \cap \mathcal{D}(\pi(V))$ *and*

$$L_V := L + \pi(V) - J\pi(V)J$$

is essentially self-adjoint on $\mathcal{D}(L) \cap \mathcal{D}(\pi(V)) \cap \mathcal{D}(J\pi(V)J)$.

Theorem 2.3 [DJP] *Assume that 2.A holds and set*

$$\tau_V^t(A) := \pi^{-1}\left(\mathrm{e}^{\mathrm{i}t(L+\pi(V))}\pi(A)\mathrm{e}^{-\mathrm{i}t(L+\pi(V)}\right).$$

Then τ_V is a W^*-dynamics on $\mathfrak{M}$ and L_V is the Liouvillean of $(\mathfrak{M}, \tau_V)$.

Our final subject is the perturbation theory of KMS states. We will describe the results of [DJP], which extend the well known results of Araki [Ar, BR2] valid for bounded perturbations.

Let ω be a (τ, β)-KMS state and $\Omega \in \mathcal{H}_+$ its vector representative. We will call Ω a (τ, β)-KMS vector (or a β-KMS-vector for the dynamics τ).

We make the following additional assumption on the perturbation V:

Assumption 2.B $\quad \|\mathrm{e}^{-\beta\pi(V)/2}\Omega\| < \infty.$

Theorem 2.4 [DJP] *Assume that Assumptions 2.A and 2.B hold. Then*
(1) *$\Omega \in \mathcal{D}(\mathrm{e}^{-\beta(L+\pi(V))/2})$ and the vector $\Omega_V := \mathrm{e}^{-\beta(L+\pi(V))/2}\Omega$ is a (τ_V, β)-KMS vector.*
(2) *Let $\omega_V(A) = (\Omega_V|\pi(A)\Omega_V)/\|\Omega_V\|^2$. Then ω_V is a (τ_V, β)-KMS state on $\mathfrak{M}$.*

Note that if $V \in \mathfrak{M}$, then Assumptions 2.A and 2.B are automatically satisfied, and the above theorems reduce to the well-known results of Araki [Ar, BR2].

3 Simplified presentation of the main results

This section gives a self-contained description of simplified versions of our main results. It will not be used in the remaining part of the paper. The reader who prefers a more complete exposition can skip this section and go directly to Section 4.

In this section the 1-particle bosonic space is $\mathcal{Z} = L^2(\mathbb{R}^d)$ and the 1-particle energy h is the operator of multiplication by $|\xi|$, where $\xi \in \mathbb{R}^d$ describes the momentum. The small system is described by a finite dimensional Hilbert space $\mathcal{K}$ and a Hamiltonian K. The interaction is described by a measurable operator-valued function (form-factor) $\mathbb{R}^d \ni \xi \mapsto v(\xi) \in \mathcal{B}(\mathcal{K})$.

3.1 Pauli-Fierz system at zero temperature

The Hilbert space of the Pauli-Fierz system at zero temperature is $\mathcal{K} \otimes \Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d))$, where $\Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d))$ denotes the symmetric (bosonic) Fock space over the 1-particle space $L^2(\mathbb{R}^d)$. The free Pauli-Fierz Hamiltonian is

$$H_{\mathrm{fr}} := K \otimes 1 + 1 \otimes \int |\xi| a^*(\xi) a(\xi) \mathrm{d}\xi,$$

where $a^*(\xi)/a(\xi)$ are the creation/annihilation operators of bosons of momentum $\xi \in \mathbb{R}^d$. We assume that the form-factor satisfies

$$\int (1+|\xi|^{-1})\|v(\xi)\|^2 \mathrm{d}\xi < \infty. \tag{3.4}$$

The interaction is given by the operator

$$V := \int (v(\xi) \otimes a^*(\xi) + v^*(\xi) \otimes a(\xi))\mathrm{d}\xi,$$

and the full Pauli-Fierz Hamiltonian equals

$$H := H_{\mathrm{fr}} + \lambda V,$$

where $\lambda \in \mathbb{R}$. H is self-adjoint on $\mathcal{D}(H_{\mathrm{fr}})$ and bounded from below.

We have discussed in [DJ1] how Pauli-Fierz Hamiltonians arise as an approximation to the standard Hamiltonian of the non-relativistic QED. A related discussions can be found in [BFS1].

A simplest non-trivial example of a Pauli-Fierz Hamiltonian is the so-called spin-boson model where $\mathcal{K} = \mathbb{C}^2$, $K = \sigma_z$ and $v(\xi) = \sigma_x \alpha(\xi)$ (σ_z and σ_x are the usual Pauli matrices and $\alpha \in L^2(\mathbb{R}^d)$ satisfies (3.4)).

3.2 Bosonic fields at non-zero density

Assume that the radiation density of the bosonic field is described by a positive measurable function $\rho(\xi)$ on $\mathbb{R}^d$. The observables of the bosonic reservoir are then described by the W^*-algebra $\mathfrak{M}_{\rho,\mathrm{l}}^{\mathrm{AW}}$, the (left) Araki-Woods algebra at density ρ. This algebra is constructed as follows. It is represented on the Hilbert space $\Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d))$. The creation/annihilation operators corresponding to the first $L^2(\mathbb{R}^d)$ (which describe excitations) are denoted by $a_{\mathrm{l}}^*(\xi)/a_{\mathrm{l}}(\xi)$, and those corresponding to the second $L^2(\mathbb{R}^d)$ (describing holes) are denoted by $a_{\mathrm{r}}^*(\xi)/a_{\mathrm{r}}(\xi)$. (l/r stand for left/right). $\mathfrak{M}_{\rho,\mathrm{l}}^{\mathrm{AW}}$ is generated by the operators of the form

$$\exp \mathrm{i} \left(\int (f(\xi)(1+\rho(\xi))^{\frac{1}{2}} a_{\mathrm{l}}^*(\xi) + \overline{f}(\xi)\rho(\xi)^{\frac{1}{2}} a_{\mathrm{r}}(\xi) + \mathrm{hc})\mathrm{d}\xi \right),$$

where $f \in L^2(\mathbb{R}^d)$ satisfies $\int |f(\xi)|^2 \rho(\xi)\mathrm{d}\xi < \infty$.

3.3 Pauli-Fierz systems at non-zero density

The Pauli-Fierz algebra at density ρ, $\mathfrak{M}_\rho$, is defined by $\mathfrak{M}_\rho := \mathcal{B}(\mathcal{K}) \otimes \mathfrak{M}_{\rho,\mathrm{l}}^{\mathrm{AW}}$. To define the dynamics, we need the following assumption:

Assumption 3.A $\int (1+|\xi|^2)(1+\rho(\xi))\|v(\xi)\|^2 \mathrm{d}\xi < \infty.$

Set

$$L_{\rm fr}^{\rm semi} := K\otimes 1 + 1\otimes \int \big(|\xi| a_{\rm l}^*(\xi)a_{\rm l}(\xi) - |\xi| a_{\rm r}^*(\xi)a_{\rm r}(\xi)\big){\rm d}\xi,$$

$$V_\rho := \int v(\xi)\otimes \big((1+\rho(\xi))^{\frac12}a_{\rm l}^*(\xi) + \rho(\xi)^{\frac12}a_{\rm r}(\xi)\big){\rm d}\xi + {\rm hc},$$

$$L_\rho^{\rm semi} := L_{\rm fr}^{\rm semi} + \lambda V_\rho. \tag{3.5}$$

Proposition 3.1 *Assume that Assumption 3.A holds. Then the operator $L_\rho^{\rm semi}$ is essentially self-adjoint on $\mathcal{D}(L_{\rm fr})\cap\mathcal{D}(V_\rho)$ and*

$$\tau_\rho^t(A) := {\rm e}^{{\rm i}tL_\rho^{\rm semi}}A{\rm e}^{-{\rm i}tL_\rho^{\rm semi}} \tag{3.6}$$

is a W^*-dynamics on $\mathfrak{M}_\rho$.

We will call the W^*-dynamical system $(\mathfrak{M}_\rho,\tau_\rho)$ the Pauli-Fierz system at density ρ. In the absence of interaction ($\lambda=0$) we call it a free Pauli-Fierz system.

The identity representation $\mathfrak{M}_\rho\to\mathcal{B}(\mathcal{K}\otimes\Gamma_{\rm s}(L^2(\mathbb{R}^d)\oplus L^2(\mathbb{R}^d))$ will be called the semistandard representation of the Pauli-Fierz system, to distinguish it from the standard representation described in the next subsection. Similarly, we will call the operator $L_\rho^{\rm semi}$ the Pauli-Fierz semi-Liouvillean at density ρ.

3.4 Pauli-Fierz systems in standard representation

Let $\overline{\mathcal{K}}$ be a Hilbert space complex conjugate to $\mathcal{K}$. The standard representation of the algebra $\mathfrak{M}_\rho$ is realized on the Hilbert space

$$\mathcal{H} := \mathcal{K}\otimes\overline{\mathcal{K}}\otimes\Gamma_{\rm s}(L^2(\mathbb{R}^d)\oplus L^2(\mathbb{R}^d)),$$

and for $B\otimes C\in\mathcal{B}(\mathcal{K})\otimes\mathfrak{M}_{\rho,{\rm l}}^{\rm AW}$,

$$\pi(B\otimes C) = B\otimes 1_{\overline{\mathcal{K}}}\otimes C.$$

For the description of the cone $\mathcal{H}_{\rho,+}$ and the modular conjugation J we refer the reader to Subsection 6.6.

Note that V_ρ is affiliated to $\mathfrak{M}_\rho$ and

$$\pi(V_\rho) = \int v(\xi)\otimes 1\otimes \big((1+\rho(\xi))^{\frac12}a_{\rm l}^*(\xi)+\rho(\xi)^{\frac12}a_{\rm r}(\xi)\big){\rm d}\xi + {\rm hc},$$

$$J\pi(V_\rho)J = \int 1\otimes\overline{v}(\xi)\otimes\big((1+\rho(\xi))^{\frac12}a_{\rm r}^*(\xi)+\rho(\xi)^{\frac12}a_{\rm l}(\xi)\big){\rm d}\xi + {\rm hc}.$$

The Liouvillean of the free Pauli-Fierz system is

$$L_{\rm fr} = K\otimes 1\otimes 1 - 1\otimes\overline{K}\otimes 1 + 1\otimes 1\otimes\int\big(|\xi|a_{\rm l}^*(\xi)a_{\rm l}(\xi) - |\xi|a_{\rm r}^*(\xi)a_{\rm r}(\xi)\big){\rm d}\xi.$$

Set

$$L_\rho := L_{\rm fr} + \lambda\pi(V_\rho) - \lambda J\pi(V_\rho)J.$$

Proposition 3.2 *Assume that Assumption (3.A) holds. Then the operator L_ρ is essentially self-adjoint on $\mathcal{D}(L_{\mathrm{fr}}) \cap \mathcal{D}(\pi(V_\rho)) \cap \mathcal{D}(J\pi(V_\rho)J)$ and is the Liouvillean of the Pauli-Fierz system $(\mathfrak{M}_\rho, \tau_\rho)$.*

Let us note that from the mathematical point of view Pauli-Fierz Hamiltonians, semi-Liouvilleans and Liouvilleans belong to the class of operators that we call Pauli-Fierz operators. This class of operators has been studied in detail in our previous paper [DJ1].

3.5 Thermal Pauli-Fierz systems

Let $\beta > 0$ be the inverse temperature. A Pauli-Fierz system whose radiation density is given by the Planck law $\rho_\beta(\xi) = (\mathrm{e}^{\beta|\xi|} - 1)^{-1}$ is called a thermal Pauli-Fierz system at inverse temperature β. Due to the specific form of the Planck law Assumption 3.A takes a somewhat simpler form and is equivalent to:

Assumption 3.B $\quad \int (|\xi|^2 + |\xi|^{-1}) \|v(\xi)\|^2 \mathrm{d}\xi < \infty.$

With a slight abuse of the notation instead of the subscript ρ_β we will use β, so L_β and τ_β^t now stand for L_{ρ_β} and $\tau_{\rho_\beta}^t$ etc. Using the main result of [DJP] described in Theorem 2.4 (see Theorem 7.3) one can easily show

Theorem 3.3 *Assume that Assumption 3.B holds. Then for all $\lambda \in \mathbb{R}$ and $\beta \in]0, \infty[$ the Pauli-Fierz system $(\mathfrak{M}_\beta, \tau_\beta)$ has a unique β-KMS state.*

3.6 Main results

In this subsection we state simplified versions of our main results, described more precisely and proved in Section 7. We use the following notation. $\mathrm{sp}(K)$ denotes the spectrum of K and $k_0 = \inf \mathrm{sp}(K)$. The spectral projection of K onto $k \in \mathbb{R}$ will be denoted by $1_k(K)$ and $v^{k_1,k_2}(\xi) = 1_{k_1}(K) v(\xi) 1_{k_2}(K)$. Obviously, $v^{k_1,k_2}(\xi) = 0$ unless $k_1, k_2 \in \mathrm{sp}(K)$. $p \in \mathbb{R}_+$ denotes the radial coordinate. S^{d-1} is the $d-1$-dimensional unit sphere, $\omega \in S^{d-1}$ is the angle coordinate and $\mathrm{d}\omega$ is the surface measure on S^{d-1}.

Let $\mathcal{F}^+$ be the set of positive differences of eigenvalues of K. (In physical terms, these are the Bohr frequencies of the small system—the energies of photons that can be emitted).

An important role will be played by a certain subset $\mathfrak{N}$ of bounded operators on $\mathcal{K}$ defined as follows: $B \in \mathcal{B}(\mathcal{K})$ belongs to $\mathfrak{N}$ iff for almost all $\omega \in S^{d-1}$ we have

$$\begin{aligned} &B \sum_{k \in \mathrm{sp}(K)} v^{k-p,k}(p\omega) = \sum_{k \in \mathrm{sp}(K)} v^{k-p,k}(p\omega) B, \qquad p \in \mathcal{F}^+, \\ &B^* \sum_{k \in \mathrm{sp}(K)} v^{k-p,k}(p\omega) = \sum_{k \in \mathrm{sp}(K)} v^{k-p,k}(p\omega) B^*, \qquad p \in \mathcal{F}^+, \\ &B \sum_{k \in \mathrm{sp}(K)} \lim_{p \downarrow 0} p^{-1/2} v^{k,k}(p\omega) = \sum_{k \in \mathrm{sp}(K)} \lim_{p \downarrow 0} p^{-1/2} v^{k,k}(p\omega) B. \end{aligned} \tag{3.7}$$

Obviously, $1 \in \mathfrak{N}$. Note also that $\mathfrak{N}$ is a $*$-algebra invariant wrt $\mathrm{e}^{\mathrm{i}tK} \cdot \mathrm{e}^{-\mathrm{i}tK}$.

We start with a result which does not hold uniformly in the temperature.

Theorem 3.4 *Suppose that Assumption 3.B holds and the following conditions are satisfied:*

(1)

$$\int \|\partial_p^3 p^{-1+d/2} \langle p\rangle^{1/2} v(p\omega)\|^2 \mathrm{d}p\mathrm{d}\omega < \infty,$$

$$\partial_p^j p^{-1+d/2} v(p\omega)\Big|_{p=0} = (-1)^j \partial_p^j p^{-1+d/2} v^*(p\omega)\Big|_{p=0}, \quad j = 0,1,2, \quad \omega \in S^{d-1}.$$

(2) $\mathfrak{N} = \mathbb{C}1$.

Then for any $0 < \beta < \infty$ there exists $\lambda_0(\beta) > 0$ such that for $0 < |\lambda| < \lambda_0(\beta)$ the Pauli-Fierz Liouvillean L_β has no singular spectrum except for a simple eigenvalue at zero. Consequently, under the above conditions the system $(\mathfrak{M}_\beta, \tau_\beta)$ has the property of return to equilibrium.

Condition (1) is our regularity assumption. Note that it allows for quite singular infrared behavior of the form-factor. For example, assume that $v(\xi)$ is smooth outside of zero and of compact support. Then (1) holds if around zero

$$v(\xi) = v_0 |\xi|^{1-d/2}, \tag{3.8}$$

where $v_0 \in \mathcal{B}(\mathcal{K})$ is self-adjoint. In particular, Theorem 3.4 applies to models derived from QED in the so-called ohmic case (see Subsection 8.1).

Condition (2) is our positive temperature effective coupling assumption. Note that it does not depend on the temperature.

The next theorem holds uniformly in the temperature.

Theorem 3.5 *Suppose that Assumption 3.B holds and the following conditions are satisfied:*

(1)

$$\int \langle p\rangle \|\partial_p^3 p^{-1+d/2} v(p\omega)\|^2 \mathrm{d}p\mathrm{d}\omega < \infty;$$

$$\int p^{-5+2j} \|\partial_p^j p^{-1+d/2} v(p\omega)\|^2 \mathrm{d}p\mathrm{d}\omega < \infty, \qquad j = 0,1,2;$$

$$\partial_p^j p^{-1+d/2} v(p\omega)\Big|_{p=0} = (-1)^j \partial_p^j p^{-1+d/2} v^*(p\omega)\Big|_{p=0}, \quad j = 0,1,2, \quad \omega \in S^{d-1}.$$

(2) $\mathfrak{N} = \mathbb{C}1$.

(3) $\dim 1_{k_0}(K) = 1$ *(the operator K has a nondegenerate smallest eigenvalue).*

(4) *There exists $c > 0$ such that for all $k \in \mathrm{sp}(K)$, $k \neq k_0$,*

$$\sum_{p>0} \int_{S^{d-1}} (v^*)^{k,k-p}(p\omega) v^{k-p,k}(p\omega) \mathrm{d}\omega \geq c 1_k(K).$$

Then for any $0 < \beta_0 < \infty$ there exists $\lambda_0 > 0$ such that for $0 < |\lambda| < \lambda_0$ and $\beta \in]\beta_0, \infty[$ the Pauli-Fierz Liouvillean L_β has no singular spectrum except for a simple eigenvalue at zero. Consequently, under the above conditions the system $(\mathfrak{M}_\beta, \tau_\beta)$ has the property of return to equilibrium.

In comparison with Theorem 3.4, in Theorem 3.5 we need two additional effective coupling assumptions (3) and (4). Note also that the regularity assumption (1) of Theorem 3.5 is much stronger than that of Theorem 3.4. For example, assume that $v(\xi)$ is smooth away from zero and of compact support. Then (1) of Theorem 3.5 holds if around zero we have $v(\xi) = v_0|\xi|^\alpha$, where $v_0 \in \mathcal{B}(\mathcal{K})$ and $\alpha > (7-d)/2$ (compare with (3.8)).

Let us mention that our formalism can be applied to non-thermal radiation densities. For instance, if the small system interacts with several reservoirs at distinct temperatures, each satisfying the conditions of Theorem 3.4, then the Liouvillean has no singular spectrum. Consequently, under these assumptions the Pauli-Fierz system has no normal states.

4 Basic notation and facts

4.1 Miscellanea

We set $\mathbb{C}_+ := \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$. Throughout the paper S^{cl} denotes the closure of a set S, so $\mathbb{C}_+^{\mathrm{cl}} = \{z \in \mathbb{C} : \mathrm{Im} z \geq 0\}$. We will use the shorthand $\langle x \rangle := (1 + x^2)^{1/2}$.

$\delta(p)$ denotes the Dirac delta at 0, $\mathcal{P}p^{-1}$ the principal value of p^{-1}, and $(p+\mathrm{i}0)^{-1} := \lim_{\epsilon \downarrow 0}(p + \mathrm{i}\epsilon)^{-1}$. We will sometimes use the so-called Sochocki formula:

$$(p + \mathrm{i}0)^{-1} = \mathcal{P}p^{-1} - \mathrm{i}\pi\delta(p).$$

4.2 Operators in Hilbert spaces

Let $\mathcal{H}$ be a Hilbert space with the scalar product $(\Psi|\Phi)$, $\Psi, \Phi \in \mathcal{H}$. $\mathcal{B}(\mathcal{H})$, $\mathcal{B}_+(\mathcal{H})$ and $U(\mathcal{H})$ denote the set of bounded, bounded positive and unitary operators on $\mathcal{H}$. $l^2(\mathcal{H})$ will denote the Hilbert space of Hilbert-Schmidt operators on $\mathcal{H}$ with the scalar product $(A|B) = \mathrm{Tr}(A^*B)$. $l^2_+(\mathcal{H})$ is the set of positive Hilbert-Schmidt operators.

If $\Psi \in \mathcal{H}$, then $|\Psi)$ and $(\Psi|$ denote respectively the operators

$$\mathbb{C} \ni \lambda \mapsto \lambda\Psi \in \mathcal{H}, \quad \mathcal{H} \ni \Phi \mapsto (\Psi|\Phi) \in \mathbb{C}.$$

Obviously, $(\Psi| := |\Psi)^*$. If $\|\Psi\| = 1$, then $|\Psi)(\Psi|$ is the orthogonal projection onto the subspace spanned by Ψ.

$\mathrm{sp}(A)$ denotes the spectrum of a closed operator A on $\mathcal{H}$. If Θ is an isolated bounded subset of $\mathrm{sp}(A)$ (closed and open in the relative topology of $\mathrm{sp}(A)$), then $1_\Theta(A)$ denotes the spectral (Riesz) projection of A onto Θ.

If A is self-adjoint and Θ is a Borel subset of $\mathbb{R}$, then $1_\Theta(A)$ denotes the spectral projection of A onto Θ. $1^{\mathrm{p}}(A)$ denotes the projection onto the subspace spanned by the eigenvectors of A. $1^{\mathrm{ac}}(A)$ and $1^{\mathrm{sc}}(A) := 1 - 1^{\mathrm{ac}}(A) - 1^{\mathrm{p}}(A)$ denote respectively the projections onto the absolutely continuous and the singular continuous part of the spectrum of A. $\mathrm{sp}_{\mathrm{p}}(A)$, $\mathrm{sp}_{\mathrm{ac}}(A)$, $\mathrm{sp}_{\mathrm{sc}}(A)$ denote respectively the point spectrum (the set of eigenvalues), the absolutely continuous spectrum and the singular continuous spectrum of A.

If $z \in \mathrm{sp}(A)$ is an isolated point of $\mathrm{sp}(A)$, or A is self-adjoint, we will write $1_z(A)$ instead of $1_{\{z\}}(A)$.

We denote the real and imaginary part of $A \in \mathcal{B}(\mathcal{H})$ by

$$A^{\mathrm{R}} := \frac{1}{2}(A + A^*), \qquad A^{\mathrm{I}} := \frac{1}{2\mathrm{i}}(A - A^*).$$

Clearly, $A = A^{\mathrm{R}} + \mathrm{i}A^{\mathrm{I}}$. A is called dissipative if $A^{\mathrm{I}} \le 0$.

4.3 Level Shift Operator

In the physics literature, the formulas for computing 2nd order corrections for eigenvalues and especially resonances often go under the name of the Fermi Golden Rule. In this subsection we will introduce an operator, sometimes called the Level Shift Operator, that can be used to formalize the Fermi Golden Rule.

Suppose that $\mathcal{H}$ is a Hilbert space with a distinguished finite dimensional subspace $\mathcal{H}^{\mathrm{v}}$. We set $\mathcal{H}^{\overline{\mathrm{v}}} := (\mathcal{H}^{\mathrm{v}})^\perp$. We will often use 2×2 matrix notation for operators on $\mathcal{H} = \mathcal{H}^{\mathrm{v}} \oplus \mathcal{H}^{\overline{\mathrm{v}}}$. For example, any $A \in \mathcal{B}(\mathcal{H})$ can be written as

$$A = \begin{bmatrix} A^{\mathrm{vv}} & A^{\mathrm{v}\overline{\mathrm{v}}} \\ A^{\overline{\mathrm{v}}\mathrm{v}} & A^{\overline{\mathrm{v}}\overline{\mathrm{v}}} \end{bmatrix}. \tag{4.9}$$

Suppose that L_{fr} is a self-adjoint operator that leaves $\mathcal{H}^{\mathrm{v}}$ invariant. Then

$$L_{\mathrm{fr}} = \begin{bmatrix} L_{\mathrm{fr}}^{\mathrm{vv}} & 0 \\ 0 & L_{\mathrm{fr}}^{\overline{\mathrm{v}}\overline{\mathrm{v}}} \end{bmatrix}. \tag{4.10}$$

For $A \in \mathcal{B}(\mathcal{H}^{\mathrm{v}})$ and $e_1, e_2 \in \mathbb{R}$ we set

$$A^{e_1,e_2} := 1_{e_1}(L_{\mathrm{fr}}^{\mathrm{vv}}) A 1_{e_2}(L_{\mathrm{fr}}^{\mathrm{vv}}).$$

Let Q be a self-adjoint operator on $\mathcal{H}$ such that $Q^{\mathrm{vv}} = 0$ and $Q^{\mathrm{v}\overline{\mathrm{v}}}$ is bounded. Let

$$w(z) := Q^{\mathrm{v}\overline{\mathrm{v}}}(z1^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - L_{\mathrm{fr}}^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1} Q^{\overline{\mathrm{v}}\mathrm{v}}, \qquad z \notin \mathrm{sp}(L_{\mathrm{fr}}^{\overline{\mathrm{v}}\overline{\mathrm{v}}}). \tag{4.11}$$

Assume that for all $e \in \mathrm{sp}(L_{\mathrm{fr}}^{\mathrm{vv}})$ the limit

$$\lim_{\epsilon \downarrow 0} w(e + \mathrm{i}\epsilon)^{ee} =: w(e + \mathrm{i}0)^{ee}$$

exists and set
$$\Gamma := \sum_{e\in \mathrm{sp}(L_{\mathrm{fr}}^{\mathrm{vv}})} w(e+\mathrm{i}0)^{ee}.$$

We will call Γ the Level Shift Operator associated to the triple $(\mathcal{H}^{\mathrm{v}}, L_{\mathrm{fr}}, Q)$.

Note that Γ is a dissipative operator (in general, Γ is not self-adjoint), $\Gamma^{e_1,e_2}=0$ for $e_1\neq e_2$, $L_{\mathrm{fr}}^{\mathrm{vv}}\Gamma^{\mathrm{R}} = \Gamma^{\mathrm{R}}L_{\mathrm{fr}}^{\mathrm{vv}}$ and $L_{\mathrm{fr}}^{\mathrm{vv}}\Gamma^{\mathrm{I}} = \Gamma^{\mathrm{I}}L_{\mathrm{fr}}^{\mathrm{vv}}$.

Let us now describe applications of the Level Shift Operator.

Assume that $L^{\overline{\mathrm{vv}}} + Q^{\overline{\mathrm{vv}}}$ is essentially self-adjoint on $\mathcal{D}(L^{\overline{\mathrm{vv}}})\cap\mathcal{D}(Q^{\overline{\mathrm{vv}}})$. Then we can define the self-adjoint operator

$$L = L_{\mathrm{fr}} + \lambda Q,$$

where $\lambda\in\mathbb{R}$ is a coupling constant. The Level Shift Operator Γ can be used to describe some properties of the operator L for a small coupling constant.

First of all, if we make appropriate analyticity assumptions similar to those of [JP1, JP2], then the operator
$$L_{\mathrm{fr}}^{\mathrm{vv}} + \lambda^2\Gamma \tag{4.12}$$
can be used to predict the approximate location and the multiplicities of eigenvalues and resonances of L for small λ.

If we do not make analyticity assumptions, then we cannot define the notion of resonance. Still, the Level Shift Operator can be used to study eigenvalues of L. In particular, in [DJ1] we proved that for a certain class of Pauli-Fierz operators L and for a small nonzero λ, the operator

$$\sum_{m\in\mathrm{sp}(\Gamma)\cap\mathbb{R}} 1_e(\Gamma)(L_{\mathrm{fr}}^{\mathrm{vv}} + \lambda^2\Gamma) \tag{4.13}$$

predicts the approximate location of eigenvalues of L and that the estimate

$$\dim 1^{\mathrm{p}}(L) \leq \dim \mathrm{Ker}\Gamma^{\mathrm{I}}$$

gives an upper bound on their multiplicity. These results will be described in Subsection 5.4.

4.4 Space $L^2(\mathbb{R})$

In this subsection we describe some operators acting on $L^2(\mathbb{R})$.

Let r denote the self-adjoint operator of multiplication by the variable in $\mathbb{R}$,

$$\big(r\Psi\big)(p) := p\Psi(p).$$

Throughout this paper, in the context of the space $L^2(\mathbb{R})$ the generic name for a variable in $\mathbb{R}$ will be p. On the other hand, the multiplication operator on $L^2(\mathbb{R})$ by its natural variable will be denoted by r.

We denote by s the self-adjoint operator

$$s\Psi(p) := -\mathrm{i}\partial_p \Psi(p), \tag{4.14}$$

and by $C(\mathbb{R})$ the set of all continuous functions on $\mathbb{R}$.

For $p \in \mathbb{R}$, define an operator $\pi_p : C(\mathbb{R}) \to \mathbb{R}$ by $\pi_p f := f(p)$. It is well known that, for $\eta > 1/2$, $\mathcal{D}(\langle s\rangle^\eta)$ is a subset of $C(\mathbb{R})$. Hence the operator $\pi_p \langle s\rangle^{-\eta}$ is well defined on $L^2(\mathbb{R})$.

The following two results are well known.

Proposition 4.1 *Let $\eta > 1/2$. Then*
(1) *The functional $\pi_p\langle s\rangle^{-\eta}$ is bounded on $L^2(\mathbb{R})$.*
(2) *The map $\mathbb{R} \ni p \mapsto \pi_p\langle s\rangle^{-\eta} \in \mathcal{B}(L^2(\mathbb{R}), \mathbb{C})$ is continuous.*

Proposition 4.2 *Let n be a positive integer and $\eta > n - \frac{1}{2}$. Then the function*

$$\mathbb{C}_+ \ni z \mapsto \langle s\rangle^{-\eta}(z-r)^{-n}\langle s\rangle^{-\eta} \in \mathcal{B}(L^2(\mathbb{R}))$$

extends from $\mathbb{C}_+$ to a continuous function on $\mathbb{C}_+^{\mathrm{cl}}$.

Let $\eta > 1/2$. In what follows, the functional $\pi_p\langle s\rangle^{-\eta}$ will be often used in the following context. Let $\mathcal{G}$ be a Hilbert space. Obviously, $\mathcal{G} \otimes \mathbb{C} = \mathcal{G}$. Hence, we can introduce the family of maps

$$1_{\mathcal{G}} \otimes \pi_p\langle s\rangle^{-\eta} : \mathcal{G} \otimes L^2(\mathbb{R}) \to \mathcal{G}. \tag{4.15}$$

Clearly, the maps (4.15) are bounded and depend continuously on p.

4.5 Space $L^2(\mathbb{R}, \mathcal{G})$

Let $\mathcal{G}$ be a Hilbert space (not necessarily separable). We say that a function $\mathbb{R} \ni p \mapsto \Psi(p) \in \mathcal{G}$ belongs to $\mathcal{L}^2(\mathbb{R}, \mathcal{G})$ iff

(1) There exists a separable subspace $\mathcal{G}_0$ such that $\Psi(p) \in \mathcal{G}_0$ for all $p \in \mathbb{R}$.
(2) For any $\Phi \in \mathcal{G}$, the function $\mathbb{R} \ni p \mapsto (\Phi|\Psi(p)) \in \mathbb{C}$ is measurable.
(3) $\int \|\Psi(p)\|^2 \mathrm{d}p < \infty$.

Let $\mathcal{N}(\mathbb{R}, \mathcal{G})$ be the set of all $\Psi \in \mathcal{L}^2(\mathbb{R}, \mathcal{G})$ such that $\Psi(p) = 0$ for almost all $p \in \mathbb{R}$ and

$$L^2(\mathbb{R}, \mathcal{G}) := \mathcal{L}^2(\mathbb{R}, \mathcal{G})/\mathcal{N}(\mathbb{R}, \mathcal{G}).$$

There exists a unique unitary operator

$$\mathcal{G} \otimes L^2(\mathbb{R}) \to L^2(\mathbb{R}, \mathcal{G}) \tag{4.16}$$

such that $\Psi \otimes f \in \mathcal{G} \otimes L^2(\mathbb{R})$ is mapped onto $p \mapsto f(p)\Psi$.

Let $\mathcal{K}$ be another Hilbert space and $q \in \mathcal{B}(\mathcal{K}, \mathcal{G} \otimes L^2(\mathbb{R}))$. Suppose that for some $\eta > 1/2$, $1_{\mathcal{G}} \otimes \langle s\rangle^\eta\, q \in \mathcal{B}(\mathcal{K}, \mathcal{G} \otimes L^2(\mathbb{R}))$. Then for $p \in \mathbb{R}$ we can define

$$q(p) := 1_{\mathcal{G}} \otimes \pi_p\; q = 1_{\mathcal{G}} \otimes \pi_p\langle s\rangle^{-\eta}\; 1_{\mathcal{G}} \otimes \langle s\rangle^\eta q \in \mathcal{B}(\mathcal{K}, \mathcal{G}).$$

Clearly

$$\mathbb{R} \ni p \mapsto q(p) \in \mathcal{B}(\mathcal{K}, \mathcal{G})$$

is a continuous function. Note that for $\Psi \in \mathcal{K}$, the vector $q\Psi \in \mathcal{G} \otimes L^2(\mathbb{R})$ can be identified through (4.16) with the function $p \mapsto q(p)\Psi$ in $L^2(\mathbb{R}, \mathcal{G})$.

For any $f \in L^\infty(\mathbb{R})$ the following identity holds:

$$q^* f(r) q = \int q^*(p) f(p) q(p) \mathrm{d}p. \tag{4.17}$$

Note the estimate

$$\|q^* q\| \leq \int \|q^*(p) q(p)\| \mathrm{d}p = \int \|q(p)\|^2 \mathrm{d}p. \tag{4.18}$$

4.6 Conjugate Hilbert spaces

Let $\mathcal{K}$ be a Hilbert space. The space $\overline{\mathcal{K}}$ conjugate to $\mathcal{K}$ is any Hilbert space with a distinguished antiunitary map

$$\mathcal{K} \ni \Psi \mapsto \overline{\Psi} \in \overline{\mathcal{K}}. \tag{4.19}$$

The map (4.19) is called the conjugation on $\mathcal{K}$.

By the Riesz lemma, the map

$$\overline{\mathcal{K}} \ni \overline{\Psi} \mapsto (\Psi| \in \mathcal{B}(\mathcal{K}, \mathbb{C})$$

is an isomorphism between $\overline{\mathcal{K}}$ and $\mathcal{B}(\mathcal{K}, \mathbb{C}) = \mathcal{K}^*$.

The inverse of the map (4.19), which is a conjugation on $\mathcal{K}$, will be denoted by the same symbol. Hence $\overline{\overline{\mathcal{K}}} = \mathcal{K}$ and $\overline{\overline{\Psi}} = \Psi$.

If $A \in \mathcal{B}(\mathcal{K})$, then $\overline{A} \in \mathcal{B}(\overline{\mathcal{K}})$ is defined by $\overline{\mathcal{K}} \ni \overline{\Psi} \mapsto \overline{A}\ \overline{\Psi} := \overline{A\Psi} \in \overline{\mathcal{K}}$.

We will often use the identification of the set of Hilbert-Schmidt operators $l^2(\mathcal{K})$ with $\mathcal{K} \otimes \overline{\mathcal{K}}$ so that $|\Phi_1)(\Phi_2| \in l^2(\mathcal{K})$ corresponds to $\Psi_1 \otimes \overline{\Psi}_2 \in \mathcal{K} \otimes \overline{\mathcal{K}}$. This identification can be sometimes confusing. To avoid misunderstanding we will try to make clear which convention we use at the moment. In particular, let us note that the following identities hold for $B \in \mathcal{B}(\mathcal{K})$ and $C \in \mathcal{K} \otimes \overline{\mathcal{K}} \simeq l^2(\mathcal{K})$:

$$B \otimes 1_{\overline{\mathcal{K}}}\ C = BC, \qquad 1_{\mathcal{K}} \otimes \overline{B}\ C = CB^*. \tag{4.20}$$

On the left hand side C is interpreted as an element of $\mathcal{K} \otimes \overline{\mathcal{K}}$ and on the right as an element of $l^2(\mathcal{K})$.

An antiunitary map κ on $\mathcal{K}$ such that $\kappa^2 = 1$ will be called an internal conjugation on $\mathcal{K}$. Note that if we have a fixed internal conjugation κ on $\mathcal{K}$, then $\mathcal{K}$ is naturally identified with $\overline{\mathcal{K}}$. Therefore, in this case we do not need to introduce $\overline{\mathcal{K}}$.

4.7 The $\star$ conjugation

Let $\mathcal{K}$ and $\mathcal{W}$ be Hilbert spaces. In this subsection we introduce a certain antilinear map $\star$ from a dense subspace of $\mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{W})$ to a dense subspace of $\mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \overline{\mathcal{W}})$.

Let $v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{W})$. We say that v is $\star$ conjugable if there exists $v^\star \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \overline{\mathcal{W}})$ such that for $\Phi, \Psi \in \mathcal{K}$ and $w \in \mathcal{W}$,

$$(\Phi \otimes w \,|\, v\Psi) = (v^\star \Phi | \Psi \otimes \overline{w}).$$

If $v^\star$ exists, then it is unique.

Remark 4.3 *Given an orthonormal basis $\{w_i \;:\; i \in I\}$ in $\mathcal{W}$, any $v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{W})$ can be decomposed as*

$$v = \sum_{i \in I} B_i \otimes |w_i), \tag{4.21}$$

where $B_i \in \mathcal{B}(\mathcal{K})$ and the sum should be understood in terms of the strong operator convergence. Note that $v^* v = \sum_{i \in I} B_i^* B_i$. It is easy to see that v is $\star$ conjugable iff $\sum_{i \in I} B_i B_i^*$ is bounded. If this is the case,

$$v^\star := \sum_{i \in I} B_i^* \otimes |\overline{w_i}).$$

and $v^{\star *} v^\star = \sum_{i \in I} B_i B_i^*$.

Proposition 4.4 *Suppose that either $\mathcal{K}$ or $\mathcal{W}$ is finite dimensional. Then all $v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{W})$ are $\star$ conjugable. Moreover, if $n := \min\big((\dim \mathcal{K})^2, \dim \mathcal{W}\big)$, then*

$$\|v^\star\| \leq \sqrt{n} \|v\|.$$

Proof. Clearly, $\dim \mathcal{B}(\mathcal{K}) = (\dim \mathcal{K})^2$. Therefore, we can choose an orthonormal system $\{w_i\}$ in $\mathcal{W}$ with at most n elements such that (4.21) is true. Now

$$\|v^\star\|^2 = \Big\| \sum_i B_i B_i^* \Big\| \leq \sum_i \|B_i B_i^*\| = \sum_i \|B_i^* B_i\| \leq n \Big\| \sum_i B_i^* B_i \Big\| = n \|v\|^2.$$

□

Remark 4.5 *If $\mathcal{W}$ and $\mathcal{K}$ are infinite dimensional, then it is easy to find an example of $v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{W})$ which is not $\star$ conjugable.*

Notation. In what follows, if ρ is an operator on $\mathcal{W}$ and $v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{W})$, we will write ρv instead of $1_\mathcal{K} \otimes \rho\, v$.

Proposition 4.6 (1) *If v is $\star$ conjugable, then so is $v^\star$; moreover, $v^{\star\star} = v$.*

(2) *If $\rho \in \mathcal{B}(\mathcal{W})$, then $(\rho v)^\star = \overline{\rho} v^\star$.*

(3) *If $B \in \mathcal{B}(\mathcal{K})$, then $(vB)^\star = B^* \otimes 1_{\overline{\mathcal{W}}}\, v^\star$.*

4.8 Coupling Hilbert-Schmidt operators

Let $\mathcal{K}$ and $\mathcal{W}$ be Hilbert spaces. In this subsection we describe some notation and identities related to the space $\mathcal{K}\otimes\overline{\mathcal{K}}\otimes\mathcal{W}\simeq l^2(\mathcal{K})\otimes\mathcal{W}$.

Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces. If $\overline{B}\in\mathcal{B}(\overline{\mathcal{K}})$ and $A\in\mathcal{B}(\mathcal{K}\otimes\mathcal{H}_1,\mathcal{K}\otimes\mathcal{H}_2)$, we define

$$\overline{B}\check{\otimes}A := \theta^{-1}\otimes 1_{\mathcal{H}_2}\ \overline{B}\otimes A\ \theta\otimes 1_{\mathcal{H}_1}\in\mathcal{B}(\mathcal{K}\otimes\overline{\mathcal{K}}\otimes\mathcal{H}_1,\mathcal{K}\otimes\overline{\mathcal{K}}\otimes\mathcal{H}_2), \tag{4.22}$$

where $\theta:\mathcal{K}\otimes\overline{\mathcal{K}}\to\overline{\mathcal{K}}\otimes\mathcal{K}$ is defined as $\theta\,\Psi_1\otimes\overline{\Psi}_2:=\overline{\Psi}_2\otimes\Psi_1$. In other words, if $C\in\mathcal{B}(\mathcal{K})$, $A\in\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$, we set

$$\overline{B}\,\check{\otimes}\,C\otimes A := C\otimes\overline{B}\otimes A.$$

We will sometimes call the operation $\check{\otimes}$ “tensoring in the middle”.

Tr denotes the trace. In the context of coupled systems Tr will be reserved for the partial trace over the space $\mathcal{K}$. To denote the partial trace over the space $\mathcal{W}$ we will use tr. Thus, if C is an operator on $\mathcal{K}\otimes\mathcal{W}$, then tr$C$ is an operator on $\mathcal{K}$.

The following propositions describe some algebraic properties of tensoring in the middle $\check{\otimes}$ and the $\star$ operation, which we will use in our computations. They can be skipped on the first reading.

Proposition 4.7 *Let $A\in l^2(\mathcal{K})$, $B\in l^2(\mathcal{K},\mathcal{K}\otimes\mathcal{W})$, $v_{\mathrm{l}}\in\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathcal{W})$, $v_{\mathrm{r}}\in\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\overline{\mathcal{W}})$ and suppose that v_{r} is $\star$ conjugable. Then the following statements hold:*

$$\begin{aligned}
1_{\overline{\mathcal{K}}}\check{\otimes}v_{\mathrm{l}}\,A &= v_{\mathrm{l}}A,\\
1_{\overline{\mathcal{K}}}\check{\otimes}v_{\mathrm{l}}^*\,B &= v_{\mathrm{l}}^*B,\\
1_{\mathcal{K}}\otimes\overline{v}_{\mathrm{r}}\,A &= A\otimes 1_{\mathcal{W}}\ v_{\mathrm{r}}^{\star},\\
1_{\mathcal{K}}\otimes\overline{v}_{\mathrm{r}}^*\,B &= \mathrm{tr}Bv_{\mathrm{r}}^{\star *},
\end{aligned} \tag{4.23}$$

where on the left we use the $\mathcal{K}\otimes\overline{\mathcal{K}}$ notation and on the right the $l^2(\mathcal{K})$ notation.

Proof. It is sufficient to prove the statement for $v_{\mathrm{l}}=C\otimes|w)$ and $v_{\mathrm{r}}=C\otimes|\overline{w})$ where $C\in\mathcal{B}(\mathcal{K})$ and $w\in\mathcal{W}$. Then

$$\begin{aligned}
1_{\overline{\mathcal{K}}}\check{\otimes}v_{\mathrm{l}} &= C\otimes 1_{\overline{\mathcal{K}}}\otimes|w),\\
1_{\overline{\mathcal{K}}}\check{\otimes}v_{\mathrm{l}}^* &= C^*\otimes 1_{\overline{\mathcal{K}}}\otimes(w|,\\
1_{\mathcal{K}}\otimes\overline{v}_{\mathrm{r}} &= 1_{\mathcal{K}}\otimes\overline{C}\otimes|w),\\
1_{\mathcal{K}}\otimes\overline{v}_{\mathrm{r}}^* &= 1_{\mathcal{K}}\otimes\overline{C}^*\otimes(w|.
\end{aligned} \tag{4.24}$$

We may also assume that $B = D \otimes |w_0)$ for some $w_0 \in \mathcal{W}$. Using (4.20) we get

$$\begin{aligned} 1_{\overline{\mathcal{K}}}\check{\otimes} v_{\rm l}\ A &= CA \otimes |w) &&= C\otimes|w)\ A, \\ 1_{\overline{\mathcal{K}}}\check{\otimes} v_{\rm l}^*\ B &= C^*D\,(w|w_0) &&= C^*\otimes(w|\ D\otimes|w_0), \\ 1_{\mathcal{K}}\otimes\overline{v}_{\rm r}\ A &= AC^* \otimes |w) &&= A\otimes 1_{\mathcal{W}}\ \big(C\otimes|\overline{w})\big)^\star, \\ 1_{\mathcal{K}}\otimes\overline{v}_{\rm r}^*\ B &= DC(w|w_0) &&= {\rm tr}\,\big(D\otimes|w_0)\ \big(C\otimes|\overline{w})\big)^{\star *}\big)\,. \end{aligned} \tag{4.25}$$

□

Proposition 4.8 *Let $A \in l^2(\mathcal{K})$, $\rho \in \mathcal{B}(\mathcal{W})$, $v_{\rm l} \in \mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\mathcal{W})$ and $v_{\rm r} \in \mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\overline{\mathcal{W}})$. Suppose that $v_{\rm l}$, $v_{\rm r}$ are $\star$ conjugable. Then*

$$1_{\mathcal{K}}\otimes\overline{v}_{\rm r}^*\ \rho\ 1_{\overline{\mathcal{K}}}\check{\otimes} v_{\rm l}\ A = {\rm tr}\rho v_{\rm l} A v_{\rm r}^{\star *} = v_{\rm l}^{\star *}\ A\otimes\overline{\rho}^*\ v_{\rm r}, \tag{4.26}$$

$$1_{\overline{\mathcal{K}}}\check{\otimes} v_{\rm l}^*\ \rho\ 1_{\mathcal{K}}\otimes\overline{v}_{\rm r}\ A = {\rm tr} v_{\rm l}^\star\ A\ v_{\rm r}^*\overline{\rho}^* = v_{\rm l}^*\ A\otimes\rho\ v_{\rm r}^\star. \tag{4.27}$$

If $v_{{\rm l},1}, v_{{\rm l},2} \in \mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\mathcal{W})$ are $\star$ conjugable, then

$$1_{\overline{\mathcal{K}}}\check{\otimes} v_{{\rm l},1}^*\ \rho\ 1_{\overline{\mathcal{K}}}\check{\otimes} v_{{\rm l},2}\ A = {\rm tr} v_{{\rm l},1}^\star v_{{\rm l},2}^{\star *}\ A\otimes\overline{\rho}^* = v_{{\rm l},1}^*\rho v_{{\rm l},2}A. \tag{4.28}$$

If $v_{{\rm r},1}, v_{{\rm r},2} \in \mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\overline{\mathcal{W}})$ are $\star$ conjugable, then

$$1_{\mathcal{K}}\otimes\overline{v}_{{\rm r},1}^*\ \rho\ 1_{\mathcal{K}}\otimes\overline{v}_{{\rm r},2}\ A = {\rm tr} A\otimes\rho\ v_{{\rm r},2}^\star v_{{\rm r},1}^{\star *} = A v_{{\rm r},2}^*\overline{\rho}^* v_{{\rm r},1}. \tag{4.29}$$

Proof. We will prove only (4.26). First note that

$$1_{\mathcal{K}}\otimes\overline{v}_{\rm r}^*\ \rho\ 1_{\overline{\mathcal{K}}}\check{\otimes} v_{\rm l}\ A = 1_{\mathcal{K}}\otimes\overline{v}_{\rm r}^\star\ \rho\ v_{\rm l} A = {\rm tr}\rho v_{\rm l} A v_{\rm r}^{\star *}. \tag{4.30}$$

We take $v_{\rm l} = C_{\rm l}\otimes|w_{\rm l})$ and $v_{\rm r} = C_{\rm r}\otimes|\overline{w}_{\rm r})$ for some $C_{\rm l}, C_{\rm r} \in \mathcal{B}(\mathcal{K})$, $w_{\rm l}, w_{\rm r} \in \mathcal{W}$. Then (4.30) is equal to

$$\begin{aligned} &{\rm tr}\Big(C_{\rm l}\otimes|\rho w_{\rm l})\ A\ \big(C_{\rm r}\otimes|\overline{w}_{\rm r})\big)^{\star *}\Big) = {\rm tr}\Big(C_{\rm l}\otimes|\rho w_{\rm l})\ A\ C_{\rm r}\otimes(w_{\rm r}|\Big) \\ &= C_{\rm l}AC_{\rm r}\,{\rm tr}\big(|\rho w_{\rm l})(w_{\rm r}|\big) = C_{\rm l}AC_{\rm r}(\overline{w}_{\rm l}|\overline{\rho}^*\overline{w}_{\rm r}) \\ &= C_{\rm l}\otimes(\overline{w}_{\rm l}|\ A\otimes\overline{\rho}^*\ C_{\rm r}\otimes|\overline{w}_{\rm r}) = \big(C_{\rm l}\otimes|w_{\rm l})\big)^{\star *}A\otimes\overline{\rho}^*\,C_{\rm r}\otimes|\overline{w}_{\rm r}) = v_{\rm l}^{\star *}A\otimes\overline{\rho}^* v_{\rm r}. \end{aligned}$$

□

5 Abstract Pauli-Fierz operators

In this section we first introduce the notation that we will use to describe the interaction of a second-quantized system with another system. Then we introduce the class of Pauli-Fierz operators. We also describe a number of results about these operators contained in the literature, especially in [DJ1], which we will use later on.

In this section we look at Pauli-Fierz operators just as a certain class of abstract self-adjoint operators. Only in the next two sections we will put them in the context of W^*-dynamical systems.

5.1 Creation/annihilation operators in coupled systems

Suppose that $\mathcal{W}$ is a Hilbert space. $\Gamma_{\rm s}(\mathcal{W}) = \oplus_{n=0}^{\infty}\Gamma_{\rm s}^{n}(\mathcal{W})$ will denote the bosonic Fock space over $\mathcal{W}$ (see eg. [BR2], also [DJ1]).

Consider another Hilbert space $\mathcal{E}$. In this subsection we discuss the formalism of the coupled system described by the Hilbert space $\mathcal{E}\otimes\Gamma_{\rm s}(\mathcal{W})$.

Let $q\in\mathcal{B}(\mathcal{E},\mathcal{E}\otimes\mathcal{W})$. The creation and annihilation operators associated to q were introduced in [DG, DJ1]. Since we will use a somewhat different notation from [DJ1], we discuss these notions in detail.

We define the creation operator $q(a^*)$ as the (unbounded) quadratic form on $\mathcal{E}\otimes\Gamma_{\rm s}(\mathcal{W})$ whose only nonzero matrix elements are between $\Psi_{n+1}\in\mathcal{E}\otimes\Gamma_{\rm s}^{n+1}(\mathcal{W})$ and $\Psi_n\in\mathcal{E}\otimes\Gamma_{\rm s}^{n}(\mathcal{W})$, for $n\geq 0$, and are equal

$$(\Psi_{n+1}|q(a^*)\Psi_n) := \sqrt{n+1}(\Psi_{n+1}|\, q\otimes 1_{\mathcal{W}}^{\otimes n}\, \Psi_n).$$

The annihilation operator $q^*(a)$ is defined as $q(a^*)^* = q^*(a)$. Note that both $q(a^*)$ and $q^*(a)$ are closed.

Remark 5.1 *In* [DJ1] $q(a^*)$ *and* $q^*(a)$ *were denoted* $a^*(q)$ *and* $a(q)$ *respectively.*

For further reference, let us note the following straightforward facts:

Proposition 5.2 (1) *Let* $\Psi\in\mathcal{E}\otimes\Gamma_{\rm s}^{n}(\mathcal{W})$. *Then the following estimates hold:*

$$\|q(a^*)\Psi\| \leq \sqrt{n+1}\|q\|\|\Psi\|, \quad \|q^*(a)\Psi\| \leq \sqrt{n}\|q\|\|\Psi\|. \tag{5.31}$$

(2) *If* $\Psi_0\in\mathcal{E}\otimes\Gamma_{\rm s}^{0}(\mathcal{W})$, *then* $q(a^*)\Psi_0 = q\Psi_0\in\mathcal{E}\otimes\Gamma_{\rm s}^{1}(\mathcal{W})$.

(3) *If* $\Psi_1\in\mathcal{E}\otimes\Gamma_{\rm s}^{1}(\mathcal{W})$, *then* $q^*(a)\Psi_1 = q^*\Psi_1\in\mathcal{E}\otimes\Gamma_{\rm s}^{0}(\mathcal{W})$.

5.2 Pauli-Fierz operators

Let $\mathcal{E}$ and $\mathcal{W}$ be as above. From now on we will always assume that $\mathcal{E}$ is finite dimensional. Let E be a self-adjoint operator on $\mathcal{E}$ and r a self-adjoint operator on $\mathcal{W}$. A self-adjoint operator on $\mathcal{H} := \mathcal{E}\otimes\Gamma_{\rm s}(\mathcal{W})$ of the form

$$L_{\rm fr} := E\otimes 1 + 1\otimes {\rm d}\Gamma(r)$$

will be called a free Pauli-Fierz operator.

For a given $q\in\mathcal{B}(\mathcal{E},\mathcal{E}\otimes\mathcal{W})$ the Pauli-Fierz interaction is defined by

$$Q = q(a^*) + q^*(a).$$

It follows from Nelson's theorem on analytic vectors that Q is essentially self-adjoint on $\mathcal{E}\otimes\Gamma_{\rm s}^{\rm fin}(\mathcal{W})$. The operator

$$L := L_{\rm fr} + \lambda Q,$$

where $\lambda\in\mathbb{R}$, will be called a Pauli-Fierz operator.

We know two sets of assumptions that guarantee the self-adjointness of Pauli-Fierz operators.

Theorem 5.3 (1) *If $r \geq 0$ and $r^{-\frac{1}{2}}q$ is bounded, then L is self-adjoint on $\mathcal{D}(L_{\mathrm{fr}})$.* (2) *If $|r|q$ is bounded, then L is essentially self-adjoint on $\mathcal{D}(L_{\mathrm{fr}}) \cap \mathcal{D}(Q)$.*

The proof of (1) can be found in Proposition 5.2 of [DJ1] and the proof of (2) in Proposition 5.1 of [DJ1]. We remark that (1) has been known for a long time, see e.g. [BFS1]. The part (2) was first proven in [JP1].

5.3 Level Shift Operator for Pauli-Fierz operators

We start by the description of a condition which plays a central role in our study. This condition was introduced [JP1, JP2] and was also used in [DJ1, M].

We assume that there exists a Hilbert space $\mathcal{G}$ and a unitary operator $U : \mathcal{W} \to L^2(\mathbb{R}) \otimes \mathcal{G}$ such that the operator UrU^* is the operator of multiplication by the variable in $\mathbb{R}$. We fix such an operator U and identify $\mathcal{W} \equiv L^2(\mathbb{R}) \otimes \mathcal{G}$. We will often make use of the self-adjoint operator $s := -\mathrm{i}\partial_p \otimes 1_{\mathcal{G}}$ introduced already in Subsection 4.4.

Let $\mathcal{H}^{\mathrm{v}} := \mathcal{E} \otimes \Gamma_{\mathrm{s}}^0(\mathcal{W})$ be the distinguished subspace of $\mathcal{H} := \mathcal{E} \otimes \Gamma_{\mathrm{s}}(\mathcal{W})$. Note that the map

$$\mathcal{E} \ni \Psi \mapsto \Psi \otimes \Omega \in \mathcal{H}^{\mathrm{v}},$$

identifies $\mathcal{E}$ with $\mathcal{H}^{\mathrm{v}}$. Likewise, the operator L_{fr} preserves the subspace $\mathcal{H}^{\mathrm{v}}$ and $L_{\mathrm{fr}}^{\mathrm{vv}}$ is identified with the operator E on $\mathcal{E}$. Note also that $Q^{\mathrm{vv}} = 0$ and $Q^{\overline{\mathrm{v}}\mathrm{v}} = q$.

For $z \in \mathbb{C}_+$ set

$$\begin{aligned} w(z) &:= Q^{\mathrm{v}\overline{\mathrm{v}}}(z1^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - L_{\mathrm{fr}}^{\overline{\mathrm{v}}\overline{\mathrm{v}}})^{-1}Q^{\overline{\mathrm{v}}\mathrm{v}} \\ &= q^*(z - E \otimes 1 - 1 \otimes r)^{-1}q. \end{aligned}$$

The next proposition follows from Proposition 4.2. (It is also a special case of Theorem 6.1 in [DJ1]).

Proposition 5.4 *Assume that $\langle s\rangle^\eta q \in \mathcal{B}(\mathcal{E}, \mathcal{E} \otimes \mathcal{W})$ for some $\eta > 1/2$. Then the function $\mathbb{C}_+ \ni z \mapsto w(z)$ extends by continuity to a continuous function on $\mathbb{C}_+^{\mathrm{cl}}$.*

Under the condition of this proposition the Level Shift Operator for the triple $(\mathcal{E} \otimes \Gamma_{\mathrm{s}}^0(\mathcal{W}), L_{\mathrm{fr}}, Q)$ is well defined and is equal to

$$\Gamma = \sum_{e_1, e_2 \in \mathrm{sp}(E)} (q^*)^{e_1, e_2}(e_1 - e_2 + \mathrm{i}0 - r)^{-1} q^{e_2, e_1}, \tag{5.32}$$

where

$$q^{e_1, e_2} := 1_{e_1}(E) \otimes 1_{\mathcal{W}}\, q\, 1_{e_2}(E).$$

5.4 Spectral theory of Pauli-Fierz operators

The following theorem is a consequence of the main results of [DJ1].

Theorem 5.5 *Let $\eta > 2$, $\epsilon > 0$, $C > 0$ and $q \in \mathcal{B}(\mathcal{E}, \mathcal{E} \otimes \mathcal{W})$ be such that:*
(a) *L is essentially self-adjoint on $\mathcal{D}(L_{\rm fr}) \cap \mathcal{D}(Q)$ for all λ.*
(b) $\|\langle s\rangle^\eta q\| \le C$.
(c) $\Gamma^{\rm I} \le -\epsilon(1 - 1_0(\Gamma^{\rm I}))$.
Then there exists $\lambda_0 > 0$, which depends on q only through η, ϵ and C, such that for $0 < |\lambda| < \lambda_0$ the following holds:
(1) $\mathrm{sp}_{\rm sc}(L) = \emptyset$.
(2) $\dim 1^{\rm p}(L) \le \dim 1_0(\Gamma^{\rm I})$.

Remark. If (c) is replaced with the condition $\Gamma^{\rm I} < -\epsilon$, then all the conclusions of the theorem hold under the weaker assumption $\eta > 1$. Moreover, in this case $1_0(\Gamma^{\rm I}) = 0$, and we conclude that L has no point spectrum.
Proof of Theorem 5.5. By Theorems 6.2, 6.3 and 6.4 of [DJ1], there exists $\lambda_0 > 0$ such that for $0 < |\lambda| < \lambda_0$, $\mathrm{sp}_{\rm sc}(L) = \emptyset$ and $\dim 1^{\rm p}(L) \le \dim 1_{\mathbb{R}}(\Gamma)$. Since Γ is a dissipative operator, $\dim 1_{\mathbb{R}}(\Gamma) \le \dim 1_0(\Gamma^{\rm I})$ (see Proposition 3.2 of [DJ1]). Hence $\dim 1^{\rm p}(L) \le \dim 1_0(\Gamma^{\rm I})$. The proofs of Theorems 6.2, 6.3 and 6.4 yield that the constant λ_0 depends only on η, ϵ and C. $\Box$

Let us note that in [DJ1] we actually proved much more than what we stated above. The following theorem, adapted from [DJ1], expresses in precise terms the intuition that the operator

$$\sum_{m \in \mathrm{sp}(\Gamma)\cap\mathbb{R}} 1_m(\Gamma)(E + \lambda^2\Gamma)$$

predicts the approximate location of eigenvalues of L and estimates from above their multiplicity.

For $x \in \mathbb{R}$ and $\epsilon > 0$, we set $I(x,\epsilon) := [x - \epsilon, x + \epsilon]$.

Theorem 5.6 *Suppose that $\|\langle s\rangle^\eta q\| < \infty$ for some $\eta > 2$, that L is essentially self-adjoint on $\mathcal{D}(L_{\rm fr}) \cap \mathcal{D}(Q)$ for all λ and let $\kappa = 1 - \eta^{-1}$. Then there exists $\lambda_0 > 0$ and $\alpha > 0$ such that for $0 < |\lambda| < \lambda_0$, the following holds:*
(1) *If $e \in \mathrm{sp}(E)$ and $m \in \mathrm{sp}(\Gamma^{ee}) \cap \mathbb{R}$, then*

$$\dim 1^{\rm p}_{I(e+\lambda^2 m,\, \alpha|\lambda|^{2+\kappa})}(L) \le \dim 1_m(\Gamma^{ee}).$$

(2)

$$\mathrm{sp}_{\rm p}(L) \subset \bigcup_{e \in \mathrm{sp}(E)} \ \bigcup_{m \in \mathbb{R}\cap\mathrm{sp}(\Gamma^{ee})} I(e + \lambda^2 m, \alpha|\lambda|^{2+\kappa}).$$

6 Pauli-Fierz systems

In this section we consider a certain class of W^*-dynamical systems that we call Pauli-Fierz systems.

Subsections 6.1-6.4 are devoted to zero temperature Pauli-Fierz systems. From the algebraic point of view they are the simplest class of Pauli-Fierz systems. Their W^*-algebras are type I factors—they are just $\mathcal{B}(\mathcal{H})$. The generators of their dynamics in the irreducible representation are given by bounded from below Pauli-Fierz operators. Such operators will be called Pauli-Fierz Hamiltonians. A Pauli-Fierz Hamiltonian is completely determined by the data $(\mathcal{K}, K, \mathcal{Z}, h, v)$, where K is the energy operator of the small system on the Hilbert space $\mathcal{K}$ and h is a positive operator describing the boson energy on the 1-particle Hilbert space $\mathcal{Z}$. The form-factor $v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{Z})$ describes the interaction.

Subsections 6.5-6.8 are devoted to Pauli-Fierz systems at density ρ, where the radiation density operator ρ is a positive operator on $\mathcal{Z}$ commuting with the 1-particle energy operator h. A Pauli-Fierz system at density ρ is uniquely determined by $(\mathcal{K}, K, \mathcal{Z}, h, v, \rho)$. The case $\rho = 0$ corresponds to zero temperature systems.

There are two representations of Pauli-Fierz systems that we will use. The first is somewhat simpler—we will call it the semistandard representation. The second one, the standard representation, is more complicated, but also more natural from the algebraic point of view.

In both representations there are certain distinguished Pauli-Fierz operators that implement the dynamics. In the semi-standard representation this operator is called the Pauli-Fierz semi-Liouvillean and is denoted L_ρ^{semi}. In the standard representation it is called the Pauli-Fierz Liouvillean and denoted L_ρ.

The main results of this section concern the Level Shift Operator for L_ρ, denoted Γ_ρ, and are described in Section 6.7. In particular, in Theorem 6.13 we give an algebraic characterization of $\mathrm{Ker}\Gamma_\rho^{\mathrm{I}}$, which will later lead to the main effective coupling assumption of our paper.

6.1 Pauli-Fierz Hamiltonians

Throughout this section we assume that K is a self-adjoint operator on a finite dimensional Hilbert space $\mathcal{K}$, h is a positive operator on a Hilbert space $\mathcal{Z}$ and $v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{Z})$. The self-adjoint operator

$$H_{\mathrm{fr}} := K \otimes 1 + 1 \otimes \mathrm{d}\Gamma(h)$$

on $\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z})$ will be called a free Pauli-Fierz Hamiltonian. The interaction is described by the self-adjoint operator

$$V = v(a^*) + v^*(a).$$

The operator

$$H := H_{\mathrm{fr}} + \lambda V,$$

where $\lambda \in \mathbb{R}$, is called a Pauli-Fierz Hamiltonian. We will need

Assumption 6.A $h^{-\frac{1}{2}}v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{Z})$.

By Theorem 5.3, Assumption 6.A implies that H is self-adjoint on $\mathcal{D}(H_{\rm fr})$ and bounded from below.

6.2 Gluing of reservoir 1-particle spaces

An important role in our paper will be played by the Hilbert space $\mathcal{Z} \oplus \overline{\mathcal{Z}}$ and the self-adjoint operator $r := h \oplus (-\overline{h})$. The following operators:

$$\mathcal{Z} \oplus \overline{\mathcal{Z}} \ni (z_1, \overline{z}_2) \mapsto \tau(z_1, \overline{z}_2) := (\overline{z}_2, z_1) \in \overline{\mathcal{Z}} \oplus \mathcal{Z}, \tag{6.33}$$

$$\mathcal{Z} \oplus \overline{\mathcal{Z}} \ni (z_1, \overline{z}_2) \mapsto \epsilon(z_1, \overline{z}_2) := (z_2, \overline{z}_1) \in \mathcal{Z} \oplus \overline{\mathcal{Z}}, \tag{6.34}$$

will be also useful. Note that τ is linear, ϵ antilinear, and $\epsilon(z_1, \overline{z}_2) = \overline{\tau(z_1, \overline{z}_2)}$.

The most important assumption that we need is the gluing condition introduced in [JP1] and further elaborated in [DJ1].

Assumption 6.B *There exists a Hilbert space $\mathcal{G}$ and a unitary $U : \mathcal{Z} \oplus \overline{\mathcal{Z}} \to L^2(\mathbb{R}) \otimes \mathcal{G}$ such that $U^* r U$ is the operator of multiplication by the variable in $\mathbb{R}$.*

In what follows we assume that Assumption 6.B holds and we identify $\mathcal{Z} \oplus \overline{\mathcal{Z}}$ with $L^2(\mathbb{R}) \otimes \mathcal{G}$ and r with the multiplication operator $(r\Psi)(p) := p\Psi(p)$.

Let us note that $\mathcal{Z}$ is identified with $L^2(\mathbb{R}_+) \otimes \mathcal{G}$ and h with $1_{[0,\infty[}(r)r$. Likewise, $\overline{\mathcal{Z}}$ is identified with $L^2(\mathbb{R}_-) \otimes \mathcal{G}$. Thus

$$(\kappa\Psi)(p) := \overline{\Psi(-p)}, \quad p \in [0, \infty[, \tag{6.35}$$

defines an antiunitary map on $\mathcal{Z}$, which satisfies $\kappa h = h\kappa$. Thanks to κ, the $\star$ operation can be viewed as a map of $\mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{Z})$ into itself.

In the expression $(v, 0)$ below we interpret 0 as an operator from $\mathcal{K}$ to $\mathcal{K} \otimes \overline{\mathcal{Z}}$. Thus,

$$(v, 0) : \mathcal{K} \mapsto \mathcal{K} \otimes L^2(\mathbb{R}) \otimes \mathcal{G},$$

where we used the identifications

$$\mathcal{K} \otimes \mathcal{Z} \oplus \mathcal{K} \otimes \overline{\mathcal{Z}} \simeq \mathcal{K} \otimes (\mathcal{Z} \oplus \overline{\mathcal{Z}}) \simeq \mathcal{K} \otimes L^2(\mathbb{R}) \otimes \mathcal{G}.$$

This operator can be also written as a function defined for almost all $p \in \mathbb{R}$ with values in $\mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{G})$:

$$(v, 0)(p) = \begin{cases} v(p), & p > 0; \\ 0, & p \leq 0. \end{cases} \tag{6.36}$$

Let s be as in (4.14). In the following assumption we have $\eta \geq 0$.

Assumption 6.C$(\eta)_0$ $\quad \langle s \rangle^{\eta}(v, 0) \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes (\mathcal{Z} \oplus \overline{\mathcal{Z}}))$.

This assumption will be used in the next section in the analysis of the Level Shift Operator for Pauli-Fierz Hamiltonians.

6.3 Level Shift Operator for Pauli-Fierz Hamiltonians

In this subsection we will calculate the Level Shift Operator for the triple $(\mathcal{K}\otimes\Gamma_{\rm s}^0(\mathcal{Z}), H_{\rm fr}, V)$. We will use the notation introduced in Subsection 4.3. In particular, we recall that

$$v^{k_1,k_2} = 1_{k_1}(K)\otimes 1_{\mathcal{Z}}\, v\, 1_{k_2}(K).$$

Let $\mathcal{F}_k := \{k - k_1 : k_1 \in \mathrm{sp}(K)\}$ be the set of allowed transition energies from the level $k \in \mathrm{sp}(K)$. The set of positive and negative transition energies from k is denoted by $\mathcal{F}_k^+ := \mathcal{F}_k \cap]0, \infty[$ and $\mathcal{F}_k^- := \mathcal{F}_k \cap] - \infty, 0[$ respectively. We also set

$$\mathcal{F} := \bigcup_{k\in \mathrm{sp}(K)} \mathcal{F}_k, \qquad \mathcal{F}^+ := \bigcup_{k\in \mathrm{sp}(K)} \mathcal{F}_k^+.$$

Let $\mathcal{H}^{\rm v} := \mathcal{K}\otimes\Gamma_{\rm s}^0(\mathcal{Z})$ and

$$\begin{aligned} w(z) &:= V^{\rm v\overline{v}}(z1^{\overline{\rm v}\overline{\rm v}} - H_{\rm fr}^{\overline{\rm v}\overline{\rm v}})^{-1} V^{\overline{\rm v}\rm v} \\ &= v^*(z - K\otimes 1 - 1\otimes h)^{-1} v. \end{aligned}$$

Proposition 6.1 *Assume that Assumption 6.C$(\eta)_0$ holds with $\eta > \frac{1}{2}$. Then the function $\mathbb{C}_+ \ni z \mapsto w(z)$ extends to a continuous function on $\mathbb{C}_+^{\rm cl}$.*

Proof. We apply the trick of "gluing non-physical free bosons" [DJ1]. Consider the extended 1-boson space $\mathcal{Z}\oplus\overline{\mathcal{Z}}$ and define the operators $r = h \oplus (-\overline{h})$ and $q = (v, 0)$. Then, for $z \in \mathbb{C}_+$,

$$v^*(z - h)^{-1} v = q^*(z - r)^{-1} q,$$

and the statement follows from Proposition 5.4. □

The Level Shift Operator, Γ, of the triple $(\mathcal{K}\otimes\Gamma_{\rm s}^0(\mathcal{Z}), H_{\rm fr}, V)$ is equal to

$$\Gamma = \sum_{k\in \mathrm{sp}(K)} \Gamma^{kk},$$

$$\Gamma^{kk} = \sum_{p\in\mathcal{F}_k} (v^*)^{k,k-p}(p + \mathrm{i}0 - h)^{-1} v^{k-p,k}.$$

With a slight abuse of notation we set $v(p) := (v, 0)(p)$ (recall (6.36)). Let $v^{k_1,k_2}(p) := 1_{k_1}(K)\otimes 1_{\mathcal{G}}\, v(p) 1_{k_2}(K)$, $(v^*)^{k_1,k_2}(p) := 1_{k_1}(K) v(p)^*\, 1_{k_2}(K)\otimes 1_{\mathcal{G}}$. The Assumption 6.C$(\eta)_0$ with $\eta > 1/2$ ensures that $v(p)$ is a continuous function (see Subsection 4.5). Hence, in particular, $v(0) = 0$. Moreover, we have

$$\begin{aligned} (\Gamma^{\rm R})^{kk} &= \sum_{p\in\mathcal{F}_k} (v^*)^{k,k-p}\mathcal{P}(p - h)^{-1} v^{k-p,k} \\ &= \sum_{p\in\mathcal{F}_k} \int (v^*)^{k,k-p}(p_1)\mathcal{P}(p - p_1)^{-1} v^{k-p,k}(p_1)\mathrm{d}p_1, \\ (\Gamma^{\rm I})^{kk} &= -\pi \sum_{p\in\mathcal{F}_k^+} (v^*)^{k,k-p}\delta(p - h) v^{k-p,k} \\ &= -\pi \sum_{p\in\mathcal{F}_k^+} (v^*)^{k,k-p}(p) v^{k-p,k}(p). \end{aligned} \tag{6.37}$$

We have described above Γ^{I} and Γ^{R} in two forms. In the first form we use the self-adjoint operator h on $\mathcal{Z}$ and a number $p \in \mathbb{R}$. Strictly speaking, neither the principal value $\mathcal{P}(p-h)^{-1}$ nor the delta function $\delta(p-h)$ are well defined as self-adjoint operators. However, within the context of (6.37), these formulas are well defined by the integral expressions using the representation of v into a direct integral with the fibers $v(p)$.

Let k_0 denote the ground state energy of K, that is, $k_0 := \inf \mathrm{sp}(K)$. For later reference, we note that the ground states of K belong to the kernel of Γ^{I}:

Proposition 6.2 $\mathrm{Ran}1_{k_0}(K) \subset \mathrm{Ker}\Gamma^{\mathrm{I}}$.

One expects that for a "generic" form-factor v, the kernel of Γ^{I} should coincide with the subspace of ground states of K. This leads to the first effective coupling assumption that we will use in our paper.

Assumption 6.D $\mathrm{Ran}1_{k_0}(K) = \mathrm{Ker}\Gamma^{\mathrm{I}}$.

Our second effective coupling assumption is that the ground state of K is simple:

Assumption 6.E $\dim \mathrm{Ran}1_{k_0}(K) = 1$.

6.4 Spectral structure of Pauli-Fierz Hamiltonians

In this subsection we formulate the main result of this paper concerning Pauli-Fierz Hamiltonians. It will say that if the interaction v is sufficiently regular and the effective coupling assumptions 6.D and 6.E hold, then the Pauli-Fierz Hamiltonian H for small nonzero coupling constants has purely absolutely continuous spectrum except possibly for a simple eigenvalue at $\inf \mathrm{sp}(H)$.

We start with the observation that an appropriate regularity of the gluing implies the self-adjointness of H.

Theorem 6.3 *Assume that Assumption 6.C$(\eta)_0$ holds with $\eta > 1/2$. Then Assumption 6.A holds.*

Proof. We use the method described in the proof of Proposition 6.1. In particular, we use the operators q and r introduced in the proof of this proposition.

By Assumption 6.C$(\eta)_0$ with $\eta > \frac{1}{2}$ and Proposition 4.2, for any $p \in \mathbb{R}$ the operator $q^*(p + \mathrm{i}0 - r)^{-1}q$ exists and is bounded. Setting $p = 0$ gives

$$q^*(\mathrm{i}0 - r)^{-1}q = v^*h^{-1}v.$$

Hence, $h^{-\frac{1}{2}}v$ is bounded. □

Now we deduce spectral information on H.

Theorem 6.4 *Suppose that Assumption 6.C$(\eta)_0$ holds with $\eta > 2$ and that Assumptions 6.D and 6.E hold. Then there exists $\lambda_0 > 0$ such that for $0 < |\lambda| < \lambda_0$ the following holds:*
(1) $\dim 1^{\mathrm{p}}(H) \leq 1$.
(2) $\mathrm{sp}_{\mathrm{sc}}(H) = \emptyset$.

Proof. We again use the method and the notation of Proposition 6.1. Extend the space $\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z})$ to the space $\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}) \otimes \Gamma_{\mathrm{s}}(\overline{\mathcal{Z}}) \simeq \mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$. The space $\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z})$ is identified with the subspace $\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}) \otimes \Gamma_{\mathrm{s}}^0(\overline{\mathcal{Z}})$. Consider the extended operators

$$\begin{aligned} L_{\mathrm{fr}} \quad &:= H_{\mathrm{fr}} \otimes 1 - 1 \otimes \mathrm{d}\Gamma(h) \quad &&\simeq K \otimes 1 + 1 \otimes \mathrm{d}\Gamma(r), \\ Q \quad &:= V \otimes 1 &&\simeq q(a^*) + q^*(a), \end{aligned}$$

and set

$$L := L_{\mathrm{fr}} + \lambda Q \;\simeq\; H \otimes 1 - 1 \otimes \mathrm{d}\Gamma(h).$$

By Theorem 6.3, H is self-adjoint on $\mathcal{D}(H_{\mathrm{fr}})$ and therefore L is self-adjoint on $\mathcal{D}(L_{\mathrm{fr}})$ (see Section 5.2 in [DJ1]). Note also that

$$\mathrm{sp}_{\mathrm{p}}(H) = \mathrm{sp}_{\mathrm{p}}(L), \quad \mathrm{sp}_{\mathrm{sc}}(H) = \mathrm{sp}_{\mathrm{sc}}(L). \tag{6.38}$$

Clearly, L is a Pauli-Fierz operator such that $\|\langle s\rangle^\eta q\| < \infty$ for $\eta > 2$. The Level Shift Operator of the triple $(\mathcal{K} \otimes \Gamma_{\mathrm{s}}^0(\mathcal{Z} \oplus \overline{\mathcal{Z}}), L_{\mathrm{fr}}, Q)$ is equal (after the obvious identification of the Hilbert spaces) to the Level Shift Operator of the triple $(\mathcal{K} \otimes \Gamma_{\mathrm{s}}^0(\mathcal{Z}), H_{\mathrm{fr}}, V)$, which we studied in the last subsection. Assumptions 6.D and 6.E yield that $\dim \mathrm{Ker}\Gamma^{\mathrm{I}} = 1$. Therefore, Theorem 5.5 implies that there exists $\lambda_0 > 0$ such that for $0 < |\lambda| < \lambda_0$ we have $\dim 1^{\mathrm{p}}(L) \leq 1, \quad \mathrm{sp}_{\mathrm{sc}}(L) = \emptyset$. By (6.38), this implies $\dim 1^{\mathrm{p}}(H) \leq 1, \quad \mathrm{sp}_{\mathrm{sc}}(H) = \emptyset$. □

Remark 6.5 *For a large class of interactions one can show that* $\dim 1_{\inf \mathrm{sp}(H)}(H) \geq 1$*, namely that Pauli-Fierz Hamiltonian H has a ground state. Results of this kind were proven in* [AH, BFS1, Ge, Sp2, Sp3]. *If to the assumptions of Theorem 6.4 we add the assumptions of the above references, then we can replace (1) with* $\dim 1^{\mathrm{p}}(H) = 1$ *and* $\mathrm{sp}_{\mathrm{p}}(H) = \inf \mathrm{sp}(H)$.

6.5 Pauli-Fierz systems of density ρ

In this subsection we introduce Pauli-Fierz W^*-dynamical systems. They will be the main subject of the remaining part of this section.

Let $\rho \geq 0$ be an operator commuting with h. It will be called the radiation density. The left Araki-Woods W^*-algebra, denoted by $\mathfrak{M}_{\rho,1}^{\mathrm{AW}}$ is defined as the W^*-subalgebra of $\mathcal{B}(\Gamma_{\mathrm{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}}))$ generated by the operators

$$\exp \mathrm{i}\Big(\big((1+\rho)^{1/2} z, \overline{\rho}^{1/2} \overline{z}\big)(a^*) + \mathrm{hc}\Big), \quad z \in \mathcal{D}(\rho^{1/2}),$$

where $(z_1, \overline{z}_2)(a^*)$ denotes the usual creation operators on $\Gamma_{\rm s}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$. The Pauli-Fierz algebra at density ρ is defined as

$$\mathfrak{M}_\rho := \mathcal{B}(\mathcal{K}) \otimes \mathfrak{M}_{\rho,{\rm l}}^{\rm AW}. \tag{6.39}$$

The identity map

$$\mathfrak{M}_\rho \to \mathcal{B}(\Gamma_{\rm s}(\mathcal{Z} \oplus \overline{\mathcal{Z}})) \tag{6.40}$$

will be called the semistandard representation of $\mathfrak{M}_\rho$. (The bosonic part of (6.40) is already standard, the part involving $\mathcal{K}$ is not—hence the name).

Proposition 6.6 *Assume that*

$$(1+\rho)^{1/2} v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{Z}). \tag{6.41}$$

Then $\overline{\rho}^{\frac{1}{2}} v^\star$ *is a bounded operator and the operators*

$$\Big((1+\rho)^{\frac{1}{2}} v, 0\Big)(a^*) + \Big(0, v^{\star *}\overline{\rho}^{\frac{1}{2}}\Big)(a), \qquad \Big(v^*(1+\rho)^{\frac{1}{2}}, 0\Big)(a) + \Big(0, \overline{\rho}^{\frac{1}{2}} v^\star\Big)(a^*),$$

which act on $\mathcal{K} \otimes \Gamma_{\rm s}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$*, are affiliated to* $\mathfrak{M}_\rho$*.*

Proof. Using Proposition 4.4 and the fact that $\mathcal{K}$ has a finite dimension we see that the boundedness of $(1+\rho)^{1/2}v$ implies the boundedness of $((1+\rho)^{1/2}v)^\star = (1+\overline{\rho})^{1/2} v^\star$. Next note that $\overline{\rho}^{1/2}(1+\overline{\rho})^{-1/2}$ is bounded. This implies the boundedness of $\overline{\rho}^{1/2} v^\star$. $\Box$

In what follows we assume (6.41) and set

$$v_\rho := ((1+\rho)^{\frac{1}{2}} v, \overline{\rho}^{\frac{1}{2}} v^\star) \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes (\mathcal{Z} \oplus \overline{\mathcal{Z}})).$$

Note that in terms of a direct integral of operators in $\mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{G})$:

$$v_\rho(p) \quad = \begin{cases} (1+\rho)^{\frac{1}{2}} v(p), & p > 0, \\ \overline{\rho}^{\frac{1}{2}} v^\star(-p), & p \leq 0. \end{cases}$$

Let

$$\begin{aligned} V_\rho \quad &:= \Big((1+\rho)^{\frac{1}{2}} v, \overline{\rho}^{\frac{1}{2}} v^\star\Big)(a^*) + \Big(v^*(1+\rho)^{\frac{1}{2}}, v^{\star *}\overline{\rho}^{\frac{1}{2}}\Big)(a) \\ &= v_\rho(a^*) + v_\rho^*(a). \end{aligned}$$

The operator V_ρ is essentially self-adjoint on the space of finite particle vectors. Moreover, it is affiliated to $\mathfrak{M}_\rho$.

The free Pauli-Fierz semi-Liouvillean is the self-adjoint operator on $\mathcal{K} \otimes \Gamma_{\rm s}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ defined as

$$\begin{aligned} L_{\rm fr}^{\rm semi} \quad &:= K \otimes 1 + 1 \otimes {\rm d}\Gamma(h \oplus -\overline{h}) \\ &= K \otimes 1 + 1 \otimes {\rm d}\Gamma(r). \end{aligned}$$

The full Pauli-Fierz semi-Liouvillean of density ρ is

$$L_\rho^{\rm semi} := L_{\rm fr}^{\rm semi} + \lambda V_\rho. \tag{6.42}$$

Let us formulate the following assumption:

Assumption 6.F$_\rho$ $\quad (1+h)(1+\rho)^{1/2}v \in \mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\mathcal{Z})$.

Theorem 6.7 (1) $\tau_{\mathrm{fr}}^t(A) := \mathrm{e}^{\mathrm{i}tL_{\mathrm{fr}}}A\mathrm{e}^{-\mathrm{i}tL_{\mathrm{fr}}}$ *is a* W^**-dynamics on* $\mathfrak{M}_\rho$.

(2) *Suppose that Assumption 6.F*$_\rho$ *hold. Then* L_ρ^{semi} *is essentially self-adjoint on* $\mathcal{D}(L_{\mathrm{fr}}^{\mathrm{semi}})\cap\mathcal{D}(V_\rho)$ *and* $\tau_\rho^t(A) := \mathrm{e}^{\mathrm{i}tL_\rho^{\mathrm{semi}}}A\mathrm{e}^{-\mathrm{i}tL_\rho^{\mathrm{semi}}}$ *is a* W^**-dynamics on* $\mathfrak{M}_\rho$.

Proof. Part (1) is obvious.

Arguing as in the proof of Proposition 6.6 we see that Assumption 6.F$_\rho$ implies that $(1+\overline{h})\overline{\rho}^{\frac{1}{2}}v^\star$ is bounded. Hence $(1+|r|)v_\rho$ is bounded. Therefore, Theorem 5.3 yields that L_ρ^{semi} is essentially self-adjoint on $\mathcal{D}(L_{\mathrm{fr}}^{\mathrm{semi}})\cap\mathcal{D}(V_\rho)$. Since V_ρ is affiliated with $\mathfrak{M}_\rho$, Theorem 3.3 in [DJP] implies that τ_ρ is a W^*-dynamics. □

$(\mathfrak{M}_\rho, \tau_\rho)$ will be called the Pauli-Fierz W^*-dynamical system at density ρ.

6.6 Standard representation of Pauli-Fierz systems

Consider the representation $\pi : \mathfrak{M}_\rho \to \mathcal{B}(\mathcal{K}\otimes\overline{\mathcal{K}}\otimes\Gamma_{\mathrm{s}}(\mathcal{Z}\oplus\overline{\mathcal{Z}}))$ defined by

$$\pi(A) := 1_{\overline{\mathcal{K}}}\check{\otimes}A, \quad A\in\mathfrak{M}_\rho = \mathcal{B}(\mathcal{K})\otimes\mathfrak{M}_{\rho,\mathrm{l}}^{\mathrm{AW}},$$

where $\check{\otimes}$ was introduced in (4.22). Clearly,

$$\pi(\mathcal{B}(\mathcal{K})\otimes\mathfrak{M}_{\rho,\mathrm{l}}^{\mathrm{AW}}) = \mathcal{B}(\mathcal{K})\otimes 1_{\overline{\mathcal{K}}}\otimes\mathfrak{M}_{\rho,\mathrm{l}}^{\mathrm{AW}}.$$

Set $J := J_{\mathcal{K}}\otimes\Gamma(\epsilon)$, where

$$J_{\mathcal{K}}\Psi_1\otimes\overline{\Psi}_2 := \Psi_2\otimes\overline{\Psi}_1, \qquad \Psi_1,\Psi_2\in\mathcal{K}, \tag{6.43}$$

and ϵ was introduced in (6.34). Note that if $A\in\mathfrak{M}_\rho$, then

$$J\pi(A)J = 1_{\mathcal{K}}\otimes\Big(1_{\overline{\mathcal{K}}}\otimes\Gamma(\tau)\ \overline{A}\ 1_{\overline{\mathcal{K}}}\otimes\Gamma(\tau)\Big),$$

where τ was introduced in (6.33). Set

$$\mathcal{H}_{\rho,+} := \{AJA\ B\otimes\Omega,\ :\ \ A\in\mathfrak{M}_\rho,\ \ B\in l_+^2(\mathcal{K})\}^{\mathrm{cl}}.$$

Proposition 6.8

$$\Big(\pi,\ \mathcal{K}\otimes\overline{\mathcal{K}}\otimes\Gamma_{\mathrm{s}}(\mathcal{Z}\oplus\overline{\mathcal{Z}}),\ J,\ \mathcal{H}_{\rho,+}\Big)$$

is a standard representation of $\mathfrak{M}_\rho$.

Set

$$L_{\mathrm{fr}} := K\otimes 1\otimes 1 - 1\otimes\overline{K}\otimes 1 + 1\otimes 1\otimes \mathrm{d}\Gamma(r).$$

Proposition 6.9 L_{fr} *is the standard Liouvillean of the free Pauli-Fierz system* $(\mathfrak{M}_\rho, \tau_{\mathrm{fr}})$.

In what follows we assume (6.41). Note that

$$\begin{aligned}\pi(V_\rho) &:= 1_{\overline{\mathcal{K}}}\check{\otimes}V_\rho\\ &= 1_{\overline{\mathcal{K}}}\check{\otimes}\Big((1+\rho)^{\frac{1}{2}}v, \overline{\rho}^{\frac{1}{2}}v^\star\Big)(a^*) + 1_{\overline{\mathcal{K}}}\check{\otimes}\Big(v^*(1+\rho)^{\frac{1}{2}}, v^{\star *}\overline{\rho}^{\frac{1}{2}}\Big)(a)\\ &= \Big(1_{\overline{\mathcal{K}}}\check{\otimes}v_\rho\Big)(a^*) + \Big(1_{\overline{\mathcal{K}}}\check{\otimes}v_\rho^*\Big)(a).\end{aligned}$$

$\pi(V_\rho)$ is essentially self-adjoint on $\mathcal{K}\otimes\overline{\mathcal{K}}\otimes\Gamma_{\rm s}^{\rm fin}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$, affiliated to $\mathcal{B}(\mathcal{K})\otimes 1_{\overline{\mathcal{K}}}\otimes\mathfrak{M}_{\rho,\rm l}^{\rm AW}$, and

$$\begin{aligned}J\pi(V_\rho)J &:= 1_{\mathcal{K}}\otimes\Big(1_{\overline{\mathcal{K}}}\otimes\Gamma(\tau)\ \overline{V_\rho}\ 1_{\overline{\mathcal{K}}}\otimes\Gamma(\tau)\Big)\\ &= 1_{\mathcal{K}}\otimes\Big(\rho^{\frac{1}{2}}\overline{v}^\star, (1+\overline{\rho})^{\frac{1}{2}}\overline{v}\Big)(a^*) + 1_{\mathcal{K}}\otimes\Big(\overline{v}^{\star *}\rho^{\frac{1}{2}}, \overline{v}^*(1+\overline{\rho})^{\frac{1}{2}}\Big)(a)\\ &= \Big(1_{\mathcal{K}}\otimes\overline{\tau v_\rho}\Big)(a^*) + \Big(1_{\mathcal{K}}\otimes\overline{v_\rho^*\tau}\Big)(a).\end{aligned}$$

Set

$$L_\rho := L_{\rm fr} + \lambda\pi(V_\rho) - \lambda J\pi(V_\rho)J. \tag{6.44}$$

Proposition 6.10 *Suppose that Assumption 6.F$_\rho$ holds. Then L_ρ is essentially self-adjoint on $\mathcal{D}(L_{\rm fr})\cap\mathcal{D}(\pi(V_\rho))\cap\mathcal{D}(J\pi(V_\rho)J)$ and is the Liouvillean of the Pauli-Fierz system $(\mathfrak{M}_\rho,\tau_\rho)$.*

Proof. The assumptions imply that

$$(1+|r|)(1_{\overline{\mathcal{K}}}\check{\otimes}v_\rho - 1_{\mathcal{K}}\otimes\overline{\tau v_\rho}) \in \mathcal{B}(\mathcal{K}\otimes\overline{\mathcal{K}}, \mathcal{K}\otimes\overline{\mathcal{K}}\otimes(\mathcal{Z}\oplus\overline{\mathcal{Z}})),$$

and the essential self-adjointness of L_ρ follows from Theorem 5.3.

The operator $L_{\rm fr}+\lambda\pi(V_\rho)$ is essentially self-adjoint on $\mathcal{D}(L_{\rm fr})\cap\mathcal{D}(\pi(V_\rho))$ and

$$\pi(\tau_\rho^t(A)) = {\rm e}^{{\rm i}t(L_{\rm fr}+\lambda\pi(V_\rho))}\pi(A){\rm e}^{-{\rm i}t(L_{\rm fr}+\lambda\pi(V_\rho))}.$$

Hence all the conditions of Theorem 2.3 are satisfied and L_ρ is the Liouvillean of $(\mathfrak{M}_\rho,\tau_\rho)$. □

6.7 Level Shift Operator for Pauli-Fierz Liouvilleans

We will see in this subsection that the Level Shift Operator of the Liouvillean L_ρ has very special algebraic properties.

Let us formulate the following family of assumptions parametrized by the radiation density operator ρ and a number $\eta\geq 0$.

Assumption 6.C$(\eta)_\rho$ $\qquad \langle s\rangle^\eta v_\rho \in \mathcal{B}(\mathcal{K}, \mathcal{K}\otimes(\mathcal{Z}\oplus\overline{\mathcal{Z}}))$.

Note that Assumption 6.C$(\eta)_0$, introduced in Subsection 6.2, is the special case of Assumption 6.C$(\eta)_\rho$ for $\rho=0$.

In this subsection we suppose that Assumption 6.C$(\eta)_\rho$ holds with $\eta > \frac{1}{2}$ and we study the Level Shift Operator, denoted Γ_ρ, of the triple $\big(\mathcal{K}\otimes\overline{\mathcal{K}}\otimes\Gamma_{\mathrm{s}}^0(\mathcal{Z}\oplus\overline{\mathcal{Z}}), L_{\mathrm{fr}}, \pi(V_\rho)-J\pi(V_\rho)J\big)$.

Define the following self-adjoint operators on $\mathcal{K}$:

$$\begin{aligned}
\Delta_\rho^{\mathrm{R}} :=& \sum_{k\in\mathrm{sp}(K)}\sum_{p\in\mathcal{F}_k}(v_\rho^*)^{k,k-p}\mathcal{P}(p-r)^{-1}v_\rho^{k-p,k}\\
=& \sum_{k\in\mathrm{sp}(K)}\sum_{p\in\mathcal{F}_k}(v^*)^{k,k-p}(1+\rho)\mathcal{P}(p-h)^{-1}v^{k-p,k}\\
&+\sum_{k\in\mathrm{sp}(K)}\sum_{p\in\mathcal{F}_k}\mathrm{tr}\ v^{k,k-p}(v^*)^{k-p,k}\rho\mathcal{P}(p+h)^{-1},
\end{aligned}$$

$$\begin{aligned}
\Delta_\rho^{\mathrm{I}} :=& -\pi\sum_{k\in\mathrm{sp}(K)}\sum_{p\in\mathcal{F}_k}(v_\rho^*)^{k,k-p}\delta(p-r)v_\rho^{k-p,k}\\
=& -\pi\sum_{k\in\mathrm{sp}(K)}\sum_{p\in\mathcal{F}_k^+\cup\{0\}}(v^*)^{k,k-p}(1+\rho)\delta(p-h)v^{k-p,k}\\
&-\pi\sum_{k\in\mathrm{sp}(K)}\sum_{p\in\mathcal{F}_k^-}\mathrm{tr}\ v^{k,k-p}(v^*)^{k-p,k}\rho\delta(p+h).
\end{aligned}$$

Set

$$\begin{aligned}
\Delta_\rho := \Delta_\rho^{\mathrm{R}} + \mathrm{i}\Delta_\rho^{\mathrm{I}} =& \sum_{k\in\mathrm{sp}(K)}\sum_{p\in\mathcal{F}_k}(v_\rho^*)^{k,k-p}(p+\mathrm{i}0-r)^{-1}v_\rho^{k-p,k}\\
=& \sum_{k\in\mathrm{sp}(K)}\sum_{p\in\mathcal{F}_k}(v^*)^{k,k-p}(1+\rho)(p+\mathrm{i}0-h)^{-1}v^{k-p,k}\\
&+\sum_{k\in\mathrm{sp}(K)}\sum_{p\in\mathcal{F}_k}\mathrm{tr}\ v^{k,k-p}(v^*)^{k-p,k}\rho(p+\mathrm{i}0+h)^{-1}.
\end{aligned}$$

For $B \in l^2(\mathcal{K})$ set also

$$\Xi_\rho(B) := 2\pi \sum_{k_1,k_2\in\mathrm{sp}(K)} \sum_{p\in\mathcal{F}_{k_1}\cap\mathcal{F}_{k_2}} (v_\rho^*)^{k_1,k_1-p}\Big(B\otimes\delta(p-r)\Big)(\tau v_\rho^\star)^{k_2-p,k_2}$$

$$= 2\pi \sum_{k_1,k_2\in\mathrm{sp}(K)} \sum_{p\in\mathcal{F}_{k_1}^+\cap\mathcal{F}_{k_2}^+\cup\{0\}} (v^*)^{k_1,k_1-p}\Big(B\otimes\delta(p-h)(1+\rho)^{\frac12}\rho^{\frac12}\Big)v^{k_2-p,k_2}$$

$$+ 2\pi \sum_{k_1,k_2\in\mathrm{sp}(K)} \sum_{p\in\mathcal{F}_{k_1}^-\cap\mathcal{F}_{k_2}^-} \mathrm{tr}\ v^{k_1,k_1-p}B(v^*)^{k_2-p,k_2}\delta(p+h)(1+\rho)^{\frac12}\rho^{\frac12}.$$

The formulas for Δ_ρ^{R}, Δ_ρ^{I}, Δ_ρ and Ξ_ρ are written in two equivalent forms. The first forms involve the operators v_ρ; the second involve v and ρ. Altough the second forms are more directly related to the basic physical quantities of interest, they are less compact and technically less convenient. Note in particular that in the formulas for Δ_ρ^{R} and Δ_ρ, the terms with $p = 0$ need to be carefully interpreted. (The singularity of $\mathcal{P}(-r)^{-1}$ and $(\mathrm{i}0-r)^{-1}$ is "cut" into two parts in these expressions. This problem is absent in the formulas involving v_ρ).

In the expression for Ξ_ρ we used the operator $\tau v_\rho^\star \in \mathcal{B}\big(\mathcal{K}, \mathcal{K}\otimes(\mathcal{Z}\oplus\overline{\mathcal{Z}})\big)$. Note that

$$\tau v_\rho^\star = \big(\rho^{\frac12}v, (1+\overline{\rho})^{\frac12}v^\star\big) = \big(\rho^{\frac12}(1+\rho)^{-\frac12}, \overline{\rho}^{-\frac12}(1+\overline{\rho})^{\frac12}\big)v_\rho, \tag{6.45}$$

$$\tau v_\rho^\star(p) = \begin{cases} \rho^{\frac12}v(p), & p\geq 0, \\ (1+\overline{\rho})^{\frac12}v^\star(-p), & p\leq 0; \end{cases}$$

Theorem 6.11 *Let $B\in l^2(\mathcal{K})$. Then*

$$\Gamma_\rho(B) = \Delta_\rho B - B\Delta_\rho^* + \mathrm{i}\Xi_\rho(B),$$

$$\Gamma_\rho^{\mathrm{R}}(B) = \Delta_\rho^{\mathrm{R}}B - B\Delta_\rho^{\mathrm{R}},$$

$$\Gamma_\rho^{\mathrm{I}}(B) = \Delta_\rho^{\mathrm{I}}B + B\Delta_\rho^{\mathrm{I}} + \Xi_\rho(B).$$

Proof. Using (5.32) we see that

$$\Gamma_\rho = \sum_{e_1,e_2\in\mathcal{F}} \big(1_{\overline{\mathcal{K}}}\check{\otimes}v_\rho^* - 1_{\mathcal{K}}\otimes\overline{v_\rho^*\tau}\big)^{e_1,e_2}(e_1-e_2+\mathrm{i}0-r)^{-1}\big(1_{\overline{\mathcal{K}}}\check{\otimes}v_\rho - 1_{\mathcal{K}}\otimes\overline{\tau v_\rho}\big)^{e_2,e_1}, \tag{6.46}$$

where the superscripts e_1, e_2 correspond to the decomposition of $\mathcal{K}\otimes\overline{\mathcal{K}}$ into the eigenspaces of $K\otimes 1 - 1\otimes\overline{K}$. (Note that $\mathrm{sp}(K\otimes 1 - 1\otimes\overline{K}) = \mathcal{F}$).

Let us switch to superscripts in $\mathrm{sp}(K)$ and to the decomposition of $\mathcal{K}$ into spectral subspaces of K. For a fixed $p \in \mathbb{R}$ we obtain

$$\begin{aligned}
\sum_{e\in\mathcal{F}} \big(1_{\overline{\mathcal{K}}}\check{\otimes} v_\rho^*\big)^{e,e-p} &= \sum_{k\in\mathrm{sp}(K)} 1_{\overline{\mathcal{K}}}\check{\otimes}(v_\rho^*)^{k,k-p},\\
\sum_{e\in\mathcal{F}} \big(1_{\mathcal{K}}\otimes\overline{v_\rho^*\tau}\big)^{e,e-p} &= \sum_{k\in\mathrm{sp}(K)} 1_{\mathcal{K}}\otimes\overline{v_\rho^*\tau}^{k-p,k},\\
\sum_{e\in\mathcal{F}} \big(1_{\overline{\mathcal{K}}}\check{\otimes} v_\rho\big)^{e-p,e} &= \sum_{k\in\mathrm{sp}(K)} 1_{\overline{\mathcal{K}}}\check{\otimes} v_\rho^{k-p,k},\\
\sum_{e\in\mathcal{F}} \big(1_{\mathcal{K}}\otimes\overline{\tau v_\rho}\big)^{e-p,e} &= \sum_{k\in\mathrm{sp}(K)} 1_{\mathcal{K}}\otimes\overline{\tau v_\rho}^{k,k-p}.
\end{aligned}$$

The terms on the right can be nonzero only if $p \in \mathcal{F}_k$. Therefore, (6.46) becomes

$$\begin{aligned}
\Gamma_\rho = \sum_{k\in\mathrm{sp}(K)} \sum_{p\in\mathcal{F}_k} \quad & \big(1_{\overline{\mathcal{K}}}\check{\otimes}(v_\rho^*)^{k,k-p}(p+\mathrm{i}0-r)^{-1}1_{\overline{\mathcal{K}}}\check{\otimes} v_\rho^{k-p,k}\\
& +1_{\mathcal{K}}\otimes\overline{v_\rho^*\tau}^{k-p,k}(p+\mathrm{i}0-r)^{-1}1_{\mathcal{K}}\otimes\overline{\tau v_\rho}^{k,k-p}\big)\\
-\sum_{k_1,k_2\in\mathrm{sp}(K)} \sum_{p\in\mathcal{F}_{k_1}\cap\mathcal{F}_{k_2}} \quad & \big(1_{\overline{\mathcal{K}}}\check{\otimes}(v_\rho^*)^{k_2,k_2-p}(p+\mathrm{i}0-r)^{-1}1_{\mathcal{K}}\otimes\overline{\tau v_\rho}^{k_1,k_1-p}\\
& +1_{\mathcal{K}}\otimes\overline{v_\rho^*\tau}^{k_1-p,k_1}(p+\mathrm{i}0-r)^{-1}1_{\overline{\mathcal{K}}}\check{\otimes} v_\rho^{k_2-p,k_2}\big).
\end{aligned}$$

Now let $B \in l^2(\mathcal{K})$. We see that $\Gamma_\rho(B)$ consists of four types of terms:
Type I Using (4.28), we obtain

$$\begin{aligned}
&\big(1_{\overline{\mathcal{K}}}\check{\otimes}(v_\rho^*)^{k,k-p}\big)(p+\mathrm{i}0-r)^{-1}\big(1_{\overline{\mathcal{K}}}\check{\otimes} v_\rho^{k-p,k}\big)B\\
&=(v_\rho^*)^{k,k-p}(p+\mathrm{i}0-r)^{-1}v_\rho^{k-p,k}B.
\end{aligned}$$

Summing up the above terms over $k \in \mathrm{sp}(K)$ and $p \in \mathcal{F}_k$ we obtain $\Delta_\rho B$.
Type II. We switch the sign in p and rename $k-p$ to k. Using first (4.29) and then $\tau\overline{r}\tau = -r$, we get

$$\begin{aligned}
&\big(1_{\mathcal{K}}\otimes(\overline{v_\rho^*\tau})^{k,k-p}\big)\big(-p+\mathrm{i}0-r\big)^{-1}\big(1_{\mathcal{K}}\otimes(\overline{\tau v_\rho})^{k-p,k}\big)B\\
&=B(v_\rho^*\tau)^{k,k-p}(-p+\mathrm{i}0-\overline{r})^{-1}\tau v_\rho^{k-p,k}\\
&=-B(v_\rho^*)^{k,k-p}(p-\mathrm{i}0-r)^{-1}v_\rho^{k-p,k}.
\end{aligned}$$

Summing up the above terms over $k \in \mathrm{sp}(K)$ and $p \in \mathcal{F}_k$ we obtain $-B\Delta_\rho^*$.
Type III. We use (4.27) to obtain

$$\begin{aligned}
&(1_{\overline{\mathcal{K}}}\check{\otimes}(v_\rho^*)^{k_2,k_2-p})(p+\mathrm{i}0-r)^{-1}(1_{\mathcal{K}}\otimes(\overline{\tau v_\rho})^{k_1,k_1-p})B\\
&=(v_\rho^*)^{k_2,k_2-p}B\otimes(p+\mathrm{i}0-r)^{-1}(\tau v_\rho^\star)^{k_1-p,k_1}.
\end{aligned}$$

Type IV. We switch the sign in p as well as rename $k_1 - p$ and $k_2 - p$ to k_1 and k_2. We use (4.26) and then $\tau\overline{r}\tau = -r$:

$$\begin{aligned}
&(1_{\mathcal{K}}\otimes(\overline{v_\rho^*\tau})^{k_1,k_1-p})(-p+\mathrm{i}0-r)^{-1}(1_{\overline{\mathcal{K}}}\check{\otimes}v_\rho^{k_2,k_2-p})B \\
&= (v_\rho^{\star *})^{k_2,k_2-p}B\otimes(-p+\mathrm{i}0-\overline{r})^{-1}\tau v_\rho^{k_1-p,k_1} \\
&= (v_\rho^{\star *}\tau)^{k_2,k_2-p}B\otimes(-p+\mathrm{i}0+r)^{-1}v_\rho^{k_1-p,k_1} \\
&= (v_\rho^{*})^{k_2,k_2-p}B\otimes(-p+\mathrm{i}0+r)^{-1}(\tau v_\rho^{\star})^{k_1-p,k_1}.
\end{aligned}$$

In the last step we used (6.45) and the fact that ρ commutes with h.

The sum of type III and IV terms over $k_1, k_2 \in \mathrm{sp}(K)$ and $p \in \mathcal{F}_{k_1} \cap \mathcal{F}_{k_2}$ equals $\mathrm{i}\Xi_\rho(B)$. $\square$

Set

$$\begin{aligned}
\widetilde{v_\rho}^{\,p} &:= \sum_{k\in\mathrm{sp}(K)} v_\rho^{k-p,k}, \qquad \widetilde{\tau v_\rho^\star}^{\,p} := \sum_{k\in\mathrm{sp}(K)} (\tau v_\rho^\star)^{k-p,k}, \\
\widetilde{v}^p &:= \sum_{k\in\mathrm{sp}(K)} v^{k-p,k}, \qquad \widetilde{v^\star}^{\,p} := \sum_{k\in\mathrm{sp}(K)} (v^\star)^{k-p,k}.
\end{aligned}$$

Here is another useful expression for Γ_ρ^{I}:

Theorem 6.12 *Let $B_1, B_2 \in l^2(\mathcal{K})$. Then*

$$-\mathrm{Tr}B_1^*\Gamma_\rho^{\mathrm{I}}(B_2) = \pi\sum_{p\in\mathcal{F}}\mathrm{Tr}\Big(\widetilde{v_\rho}^{\,p}B_1 - B_1\otimes 1\,\widetilde{\tau v_\rho^\star}^{\,p}\Big)^*\delta(p-r)\Big(\widetilde{v_\rho}^{\,p}B_2 - B_2\otimes 1\,\widetilde{\tau v_\rho^\star}^{\,p}\Big)$$

$$= \pi\sum_{p\in\mathcal{F}^+\cup\{0\}}\mathrm{Tr}\Big((1+\rho)^{\frac12}\widetilde{v}^pB_1 - B_1\otimes 1\,\rho^{\frac12}\widetilde{v}^p\Big)^*\delta(p-h)\Big((1+\rho)^{\frac12}\widetilde{v}^pB_2 - B_2\otimes 1\,\rho^{\frac12}\widetilde{v}^p\Big)$$

$$+\,\pi\sum_{p\in\mathcal{F}^+}\mathrm{Tr}\Big((1+\rho)^{\frac12}\widetilde{v}^pB_2^* - B_2^*\otimes 1\,\rho^{\frac12}\widetilde{v}^p\Big)^*\delta(p-h)\Big((1+\rho)^{\frac12}\widetilde{v}^pB_1^* - B_1^*\otimes 1\,\rho^{\frac12}\widetilde{v}^p\Big).$$

Proof. Recall that

$$-\Delta_\rho^{\mathrm{I}} = \pi\sum_{k,p}(v_\rho^*)^{k,k-p}\delta(p-r)v_\rho^{k-p,k}.$$

Hence

$$\begin{aligned}
-\mathrm{Tr}B_1^*\Delta_\rho^{\mathrm{I}}B_2 &= \pi\sum_{k,p}\mathrm{Tr}\big(v_\rho^{k-p,k}B_1\big)^*\delta(p-r)v_\rho^{k-p,k}B_2 \\
&= \pi\sum_{k_1,k_2,p}\mathrm{Tr}\big(v_\rho^{k_1-p,k_1}B_1\big)^*\delta(p-r)(v_\rho^{k_2-p,k_2}B_2).
\end{aligned}$$

There is an alternative formula for $-\Delta_\rho^{\rm I}$, which follows from $\tau\overline{r}\tau=-r$:

$$\begin{aligned}
-\Delta_\rho^{\rm I} &= \pi\sum_{k,p}{\rm tr}(v_\rho^\star)^{k,k-p}(v_\rho^{\star *})^{k-p,k}\delta(p-\overline{r})\\
&= \pi\sum_{k,p}{\rm tr}(\tau v_\rho^\star)^{k,k-p}(v_\rho^{\star *}\tau)^{k-p,k}\delta(p+r)\\
&= \pi\sum_{k,p}{\rm tr}(\tau v_\rho^\star)^{k-p,k}(v_\rho^{\star *}\tau)^{k,k-p}\delta(p-r).
\end{aligned}$$

Hence

$$\begin{aligned}
-{\rm Tr}B_1^*B_2\Delta_\rho^{\rm I} &= \pi\sum_{k,p}{\rm Tr}\big(B_1(\tau v_\rho^\star)^{k-p,k}\big)^*\delta(p-r)B_2(\tau v_\rho^\star)^{k-p,k}\\
&= \pi\sum_{k_1,k_2,p}{\rm Tr}\big(B_1(\tau v_\rho^\star)^{k_1-p,k_1}\big)^*\delta(p-r)B_2(\tau v_\rho^\star)^{k_2-p,k_2}.
\end{aligned}$$

Recall that

$$\tfrac12\Xi_\rho(B_2)=\pi\sum_{k_1,k_2,p}(v_\rho^*)^{k_1,k_1-p}\,B_2\otimes\delta(p-r)\,(\tau v_\rho^\star)^{k_2-p,k_2}. \tag{6.47}$$

Terms coming from Ξ_ρ we split as

$${\rm Tr}B_1^*\Xi_\rho(B_2)=\frac12{\rm Tr}B_1^*\Xi_\rho(B_2)+\frac12{\rm Tr}B_1^*\Xi_\rho(B_2). \tag{6.48}$$

The first term on the right of (6.48) we treat as follows:

$$\tfrac12{\rm Tr}B_1^*\Xi_\rho(B_2)=\pi\sum_{k_1,k_2,p}{\rm Tr}\big(v_\rho^{k_1-p,k_1}B_1\big)^*\delta(p-r)\,B_2(\tau v_\rho^\star)^{k_2-p,k_2}.$$

Then we transform the formula (6.47), using (6.45), (4.28) and then $\tau\overline{r}\tau=-r$:

$$\begin{aligned}
\frac12\Xi_\rho(B_2) &= \pi\sum_{k_1,k_2,p}(v_\rho^{\star *}\tau)^{k_1,k_1-p}\,B_2\otimes\delta(p-r)\,v_\rho^{k_2-p,k_2}\\
&= \pi\sum_{k_1,k_2,p}{\rm tr}(\tau v_\rho)^{k_1,k_1-p}\,B_2\,(v_\rho^{\star *})^{k_2-p,k_2}\delta(p-\overline{r})\\
&= \pi\sum_{k_1,k_2,p}{\rm tr}v_\rho^{k_1,k_1-p}\,B_2\,(v_\rho^{\star *}\tau)^{k_2-p,k_2}\delta(p+r)\\
&= \pi\sum_{k_1,k_2,p}{\rm tr}v_\rho^{k_1-p,k_1}\,B_2\,(v_\rho^{\star *}\tau)^{k_2-p,k_2}\delta(p-r).
\end{aligned}$$

Hence the second term in (6.48) also has the form

$$\tfrac{1}{2}\mathrm{Tr}B_1^*\Xi_\rho(B_2) = \pi\sum_{k_1,k_2,p}\mathrm{Tr}\big(B_1(\tau v_\rho^\star)^{k_2-p,k_2}\big)^*\delta(p-r)\,B_2 v_\rho^{k_1-p,k_1}.$$

This ends the proof of the first identity of our theorem.

Let us prove the second identity. We have

$$\begin{aligned}
&\pi\sum_{p\in\mathcal{F}}\mathrm{Tr}\Big(\widetilde{v_\rho}^p B_1 - B_1{\otimes}1\,\widetilde{\tau v_\rho^\star}^p\Big)^*\delta(p-r)\Big(\widetilde{v_\rho}^p B_2 - B_2{\otimes}1\,\widetilde{\tau v_\rho^\star}^p\Big)\\
&=\pi\sum_{p\in\mathcal{F}^+\cup\{0\}}\mathrm{Tr}\Big((1+\rho)^{\frac12}\widetilde{v}^p B_1 - B_1{\otimes}1\,\rho^{\frac12}\widetilde{v}^p\Big)^*\delta(p-h)\Big((1+\rho)^{\frac12}\widetilde{v}^p B_2 - B_2{\otimes}1\,\rho^{\frac12}\widetilde{v}^p\Big)\\
&+\pi\sum_{-p\in\mathcal{F}^+}\mathrm{Tr}\Big(\overline{\rho}^{\frac12}\widetilde{v^\star}^p B_1 - B_1{\otimes}1\,(1+\overline{\rho})^{\frac12}\widetilde{v^\star}^p\Big)^*\delta(p+\overline{h})\Big(\overline{\rho}^{\frac12}\widetilde{v^\star}^p B_2 - B_2{\otimes}1\,(1+\overline{\rho})^{\frac12}\widetilde{v^\star}^p\Big).
\end{aligned}$$

The second term on the right side can be transformed into

$$\begin{aligned}
&\pi\sum_{p\in\mathcal{F}^+}\mathrm{Tr}\Big(\overline{\rho}^{\frac12}(\widetilde{v}^p)^\star B_1 - B_1{\otimes}1\,(1+\overline{\rho})^{\frac12}(\widetilde{v}^p)^\star\Big)^*\delta(p-\overline{h})\Big(\overline{\rho}^{\frac12}(\widetilde{v}^p)^\star B_2 - B_2{\otimes}1\,(1+\overline{\rho})^{\frac12}(\widetilde{v}^p)^\star\Big)\\
&=\pi\sum_{p\in\mathcal{F}^+}\mathrm{Tr}\Big(B_1^*{\otimes}1\,\rho^{\frac12}\widetilde{v}^p - (1+\rho)^{\frac12}\widetilde{v}^p B_1^*\Big)^{\star *}\delta(p-\overline{h})\Big(B_2^*{\otimes}1\,\rho^{\frac12}\widetilde{v}^p - (1+\rho)^{\frac12}\widetilde{v}^p B_2^*\Big)^\star\\
&=\pi\sum_{p\in\mathcal{F}^+}\mathrm{Tr}\Big(B_1^*{\otimes}1\,\rho^{\frac12}\widetilde{v}^p - (1+\rho)^{\frac12}\widetilde{v}^p B_1^*\Big)\Big(B_2^*{\otimes}1\rho^{\frac12}\widetilde{v}^p - (1+\rho)^{\frac12}\widetilde{v}^p B_2^*\Big)^*\delta(p-h)\\
&=\pi\sum_{p\in\mathcal{F}^+}\mathrm{Tr}\Big((1+\rho)^{\frac12}\widetilde{v}^p B_2^* - B_2^*{\otimes}1\,\rho^{\frac12}\widetilde{v}^p\Big)^*\delta(p-h)\Big((1+\rho)^{\frac12}\widetilde{v}^p B_1^* - B_1^*{\otimes}1\,\rho^{\frac12}\widetilde{v}^p\Big).
\end{aligned}$$

In the first step we used $\widetilde{v^\star}^{-p} = (\widetilde{v}^p)^\star$, then Proposition 4.6, Proposition 4.8 and in the last step we used the cyclicity of trace. □

Since

$$\widetilde{v_\rho}^p(p) = \begin{cases}(1+\rho)^{\frac12}\widetilde{v}^p(p), & p\geq 0\\ \overline{\rho}^{\frac12}\widetilde{v^\star}^p(-p), & p\leq 0;\end{cases}$$

$$\widetilde{\tau v_\rho^\star}^p(p) = \begin{cases}\rho^{\frac12}\widetilde{v}^p(p), & p\geq 0\\ (1+\overline{\rho})^{\frac12}\widetilde{v^\star}^p(-p), & p\leq 0,\end{cases}$$

the identity proven in Theorem 6.12 can be also written as

$$
\begin{aligned}
-\mathrm{Tr}B_1^*\Gamma_\rho^{\mathrm{I}}(B_2) &= \pi\sum_{p\in\mathcal{F}}\mathrm{Tr}\Big(\widetilde{v_\rho}^p(p)\,B_1 - B_1\otimes 1\,\widetilde{\tau v_\rho^\star}^p(p)\Big)^*\Big(\widetilde{v_\rho}^p(p)\,B_2 - B_2\otimes 1\,\widetilde{\tau v_\rho^\star}^p(p)\Big)\\
&= \pi\sum_{p\in\mathcal{F}^+\cup\{0\}}\mathrm{Tr}\Big((1+\rho)^{\frac12}\widetilde{v}^p(p)\,B_1 - B_1\otimes 1\,\rho^{\frac12}\widetilde{v}^p(p)\Big)^*\\
&\quad\times\Big((1+\rho)^{\frac12}\widetilde{v}^p(p)B_2 - B_2\otimes 1\,\rho^{\frac12}\widetilde{v}^p(p)\Big)\\
&\quad+\pi\sum_{p\in\mathcal{F}^+}\mathrm{Tr}\Big((1+\rho)^{\frac12}\widetilde{v}^p(p)\,B_2^* - B_2^*\otimes 1\,\rho^{\frac12}\widetilde{v}^p(p)\Big)^*\\
&\quad\times\Big((1+\rho)^{\frac12}\widetilde{v}^p(p)\,B_1^* - B_1^*\otimes 1\,\rho^{\frac12}\widetilde{v}^p(p)\Big).
\end{aligned}
\tag{6.49}
$$

We are now ready to state the main result of this subsection.

Theorem 6.13 *$B\in\mathrm{Ker}\Gamma_\rho^{\mathrm{I}}$ iff the following two commutation relations hold:*

$$
\begin{aligned}
(1+\rho)^{\frac12}\widetilde{v}^p(p)B &= B\otimes 1\,\rho^{\frac12}\widetilde{v}^p(p), \quad p\in\mathcal{F}^+\cup\{0\},\\
(1+\rho)^{\frac12}\widetilde{v}^p(p)B^* &= B^*\otimes 1\,\rho^{\frac12}\widetilde{v}^p(p), \quad p\in\mathcal{F}^+.
\end{aligned}
\tag{6.50}
$$

Proof. Note that $B\in\mathrm{Ker}\Gamma_\rho^{\mathrm{I}}$ iff $\mathrm{Tr}B^*\Gamma_\rho^{\mathrm{I}}(B)=0$. Hence $B\in\mathrm{Ker}\Gamma_\rho^{\mathrm{I}}$ iff all the terms of (6.49) with $B_1=B_2=B$ are zero, and this is precisely the condition (6.50). □

6.8 Pauli-Fierz systems with several reservoirs

Let us describe our formalism in the case of a small system coupled to several independent reservoirs.

Let $\mathcal{Z}_i$ be Hilbert spaces and h_i, ρ_i, positive commuting self-adjoint operators on $\mathcal{Z}_i$, $i=1,\dots,n$. $\mathcal{Z}_i$ and h_i are the single particle space and energy operator of the i-th reservoir and ρ_i is the corresponding radiation density. Let $v_i\in\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathcal{Z}_i)$ be the form factor describing interaction of the small system with the i-th reservoir. Set

$$
\mathcal{Z} := \bigoplus_{i=1}^n \mathcal{Z}_i,\qquad h := \bigoplus_{i=1}^n h_i,\qquad \rho := \bigoplus_{i=1}^n \rho_i.
$$

If we impose Assumption 6.F$_\rho$ (which is equivalent to imposing Assumptions 6.F$_{\rho_i}$ for all i), then the corresponding composite Pauli-Fierz system is well-defined.

If we impose Assumptions 6.C$(\eta)_{\rho_i}$ with $\eta > \frac{1}{2}$ on v_i for all i, then the Level Shift Operator of the composite system, Γ_ρ, is well-defined and equal to

$$\Gamma_\rho = \sum_{i=1}^{n} \Gamma_{i,\rho_i}, \tag{6.51}$$

where Γ_{i,ρ_i} is the Level Shift Operator of the i-th subsystem.

7 Thermal Pauli-Fierz systems

Given a Pauli-Fierz Hamiltonian we can define a family of Pauli-Fierz systems parametrized by $\beta \in]0, \infty]$ whose radiation density is given by the Planck law at the inverse temperature β. Such systems will be called thermal Pauli-Fierz systems. They are particularly important from the physical point of view and enjoy special mathematical properties.

7.1 Thermal Pauli-Fierz Liouvilleans

The setup of this section is very similar to the setup of the previous section. In particular, the operators K, h, v and H, as well as the spaces $\mathcal{K}$ and $\mathcal{Z}$ are such as those introduced in Subsection 6.1.

Let $0 < \beta \leq \infty$. In this section we consider the family of densities

$$\rho_\beta := (e^{\beta h} - 1)^{-1}, \quad \rho_\infty = 0.$$

Note that

$$1 + \rho_\beta = (1 - \mathrm{e}^{-\beta h})^{-1} = \mathrm{e}^{\beta h} \rho_\beta, \quad 1 + \rho_\infty = 1. \tag{7.52}$$

We change slightly the notation by replacing the subscripts ρ_β with β. For instance we will write v_β, L_β, L_β^{semi}, $\mathfrak{M}_{\beta,1}^{\mathrm{AW}}$, $\mathfrak{M}_\beta$ and τ_β^t instead of v_{ρ_β}, L_{ρ_β}, $L_{\rho_\beta}^{\mathrm{semi}}$, $\mathfrak{M}_{\rho_\beta,1}^{\mathrm{AW}}$, $\mathfrak{M}_{\rho_\beta}$ and $\tau_{\rho_\beta}^t$. We warn the reader that the density $\rho = 0$ corresponds now to the inverse temperature $\beta = \infty$.

Note that

$$\begin{aligned} v_\beta &= |1 - \mathrm{e}^{-\beta r}|^{-\frac{1}{2}} (v, v^\star), \\ \overline{\tau v_\beta} &= |1 - \mathrm{e}^{\beta r}|^{-\frac{1}{2}} (\overline{v^\star}, \overline{v}), \\ \tau v_\beta^\star &= |1 - \mathrm{e}^{\beta r}|^{-\frac{1}{2}} (v, v^\star) = \mathrm{e}^{-\beta r/2} v_\beta. \end{aligned}$$

Assumption 7.A $\qquad (h^{-1/2} + h) v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{Z})$

Proposition 7.1 *Suppose Assumption 7.A holds. Then for any* $0 < \beta \leq \infty$ *Assumption 6.F*$_\rho$ *for* $\rho = \rho_\beta$ *holds.*

Proof. Clearly, for $0 < \beta \leq \infty$, $(h^{-1/2} + h)^{-1}(1+h)(1-\mathrm{e}^{-\beta h})^{-1/2}$ is bounded. Therefore Assumption 7.A implies the boundedness of $(1+h)(1-\mathrm{e}^{-\beta h})^{-1/2}v$. □

The following theorem follows immediately from Proposition 7.1 and Theorem 6.10.

Theorem 7.2 *Suppose that Assumption 7.A holds. Then, for any $0 < \beta \leq \infty$, L_β is essentially self-adjoint on $\mathcal{D}(L_{\mathrm{fr}}) \cap \mathcal{D}(\pi(V_\beta)) \cap \mathcal{D}(J\pi(V_\beta)J)$, the thermal Pauli-Fierz system $(\mathfrak{M}_\beta, \tau_\beta)$ is well defined and L_β is its Liouvillean.*

7.2 KMS states for thermal Pauli-Fierz systems

In this subsection we describe results concerning the existence of KMS states for Pauli-Fierz systems at positive temperatures. Since Pauli-Fierz W^*-algebras are factors, a thermal Pauli-Fierz system may have at most one KMS-state.

For $0 < \beta < \infty$, set

$$\gamma_\beta := \mathrm{e}^{-\beta K/2}/\sqrt{\mathrm{Tr}\ \mathrm{e}^{-\beta K}}. \tag{7.53}$$

(γ_β is the β-KMS vector of the small system). The free Pauli-Fierz system $(\mathfrak{M}_\beta, \tau_{\mathrm{fr}})$ has a unique β-KMS state and $\gamma_\beta \otimes \Omega$ is the corresponding β-KMS vector. Obviously, $\gamma_\beta \otimes \Omega \in \mathrm{Ker} L_{\mathrm{fr}}$.

The existence of KMS state for interacting Pauli-Fierz systems, which is a somewhat delicate problem because perturbation V_β is not a bounded operator and Araki's theory [Ar, BR2] cannot be applied directly, follows from the result of [DJP], reviewed in Section 2.

Theorem 7.3 *Suppose that Assumption 7.A holds and that $0 < \beta < \infty$. Then the thermal Pauli-Fierz system $(\mathfrak{M}_\beta, \tau_\beta)$ has a unique β-KMS state. Moreover, $\gamma_\beta \otimes \Omega \in \mathcal{D}(\mathrm{e}^{-\beta(L+\lambda\pi(V_\beta))/2})$ and the vector*

$$\mathrm{e}^{-\beta(L+\lambda\pi(V_\beta))/2}\gamma_\beta \otimes \Omega \tag{7.54}$$

is the β-KMS vector for $(\mathfrak{M}_\beta, \tau_\beta)$. This vector belongs to $\mathrm{Ker} L_\beta$.

Proof. By Theorem 2.4 (see also [DJP]), we need only to check that for all $\lambda \in \mathbb{R}$,

$$\|\mathrm{e}^{-\lambda\beta\pi(V_\beta)/2}\gamma_\beta \otimes \Omega\| < \infty.$$

To verify this, it suffices to show that there exists a constant c such that for all n,

$$\|\pi(V_\beta)^n\, \gamma_\beta \otimes \Omega\| \leq c^n \sqrt{(n+1)!}. \tag{7.55}$$

Since

$$\pi(V_\beta) = 1_{\overline{\mathcal{K}}} \check{\otimes} v_\beta(a^*) + 1_{\overline{\mathcal{K}}} \check{\otimes} v_\beta^*(a)$$

we can decompose $\pi(V_\beta)^n$ into the sum of 2^n-terms, each of which is a product of creation and annihilation operators. Applying the estimates (5.31) to each term we derive that (7.55) holds with $c = 2\|v_\beta\|$. □

7.3 Level Shift Operator for thermal Pauli-Fierz Liouvilleans

In this subsection we consider the Level Shift Operator in the context of thermal Liouvilleans.

Let us formulate the following family of assumptions parametrized by $\eta \geq 0$.

Assumption 7.B(η) $\qquad \langle s\rangle^\eta |r|^{-1/2}\langle r\rangle^{1/2}(v, v^\star) \in \mathcal{B}(\mathcal{K}, \mathcal{K}\otimes(\mathcal{Z}\oplus\overline{\mathcal{Z}}))$.

Proposition 7.4 *Suppose that Assumption 7.B(η) holds. Then $\langle s\rangle^\eta v_\beta \in \mathcal{B}(\mathcal{K}, \mathcal{K}\otimes(\mathcal{Z}\oplus\overline{\mathcal{Z}}))$ for all $\beta \in]0,\infty[$.*

Proof. It is easy to see that the function

$$\mathbb{R} \ni p \mapsto |1-\mathrm{e}^{-\beta p}|^{-\frac{1}{2}}\langle p\rangle^{-1/2}|p|^{1/2} \in \mathbb{R} \tag{7.56}$$

is bounded with all bounded derivatives. Hence the operator

$$\langle s\rangle^\eta |1-\mathrm{e}^{-\beta r}|^{-\frac{1}{2}}\langle r\rangle^{-1/2}|r|^{1/2}\langle s\rangle^{-\eta}$$

is bounded for all $\eta \geq 0$. Therefore, the boundedness of $\langle s\rangle^\eta |r|^{-1/2}\langle r\rangle^{1/2}(v, v^\star)$ implies the boundedness of $\langle s\rangle^\eta |1-\mathrm{e}^{-\beta r}|^{-\frac{1}{2}}(v, v^\star)$. □

The following proposition gives a condition which is easy to verify in practice and which implies Assumption 7.B(η):

Proposition 7.5 *Let n be a non-negative integer and assume that*

$$\begin{aligned}
&\int_0^\infty \|\partial_p^n p^{-1/2}\langle p\rangle^{1/2} v(p)\|^2 \mathrm{d}p < \infty;\\
&\partial_p^j p^{-1/2} v(p)\Big|_{p=0} = (-1)^j \partial_p^j p^{-1/2} v^\star(p)\Big|_{p=0}, \quad j = 0, \dots, n-1.
\end{aligned}$$

Then Assumption 7.B(n) holds.

Proof. See Proposition 7.15, which has a similar proof. □

Throughout this subsection we assume that Assumption 7.B(η) holds with $\eta > \frac{1}{2}$. We will describe the Level Shift Operator in the case $\rho = \rho_\beta$, which, consistently with our notation, will be denoted Γ_β. A special attention needs to be devoted to the infrared term in Γ_β.

Proposition 7.6 *There exists*

$$v_{\mathrm{ir}} := \lim_{p\downarrow 0} \frac{v(p)}{p^{\frac{1}{2}}} = \lim_{p\downarrow 0} \frac{v^\star(p)}{p^{\frac{1}{2}}}. \tag{7.57}$$

Set $\widetilde{v_{\mathrm{ir}}}^0 := \sum_p v_{\mathrm{ir}}^{pp}$. We have

$$v_{\mathrm{ir}} = (v_{\mathrm{ir}})^\star, \quad (\widetilde{v_{\mathrm{ir}}}^0)^\star = \widetilde{v_{\mathrm{ir}}}^0. \tag{7.58}$$

Moreover, $\Gamma^{\mathrm{I}}_\beta$ acts as follows:

$$
\begin{aligned}
-\mathrm{Tr}B_1^*\Gamma_\beta^{\mathrm{I}}(B_2) \quad &= \pi\sum_{p\in\mathcal{F}+}|\mathrm{e}^{\beta p}-1|^{-1}\mathrm{Tr}\Big(\mathrm{e}^{\beta p/2}\widetilde{v}^p(p)B_1 - B_1\otimes 1\ \widetilde{v}^p(p)\Big)^*\\
&\qquad\times\Big(\mathrm{e}^{\beta p/2}\widetilde{v}^p(p)B_2 - B_2\otimes 1\ \widetilde{v}^p(p)\Big)\\
&+\pi\sum_{p\in\mathcal{F}+}|\mathrm{e}^{-\beta p}-1|^{-1}\mathrm{Tr}\Big(\widetilde{v}^p(p)B_2^* - \mathrm{e}^{-\beta p/2}B_2^*\otimes 1\ \widetilde{v}^p(p)\Big)^*\\
&\qquad\times\Big(\widetilde{v}^p(p)B_1^* - \mathrm{e}^{-\beta p/2}B_1^*\otimes 1\ \widetilde{v}^p(p)\Big)\\
&+\tfrac{\pi}{\beta}\mathrm{Tr}\Big(\widetilde{v_{\mathrm{ir}}}^0 B_1 - B_1\otimes 1\ \widetilde{v_{\mathrm{ir}}}^0\Big)^*\\
&\qquad\times\Big(\widetilde{v_{\mathrm{ir}}}^0 B_2 - B_2\otimes 1\ \widetilde{v_{\mathrm{ir}}}^0\Big).
\end{aligned}
\tag{7.59}
$$

Proof. For $p>0$,

$$
v_\beta(p) = |1-\mathrm{e}^{-\beta p}|^{-\frac{1}{2}}v(p),
$$

$$
v_\beta(-p) = |1-\mathrm{e}^{\beta p}|^{-\frac{1}{2}}v^\star(p).
$$

Since the function $\mathbb{R}\ni p\mapsto v_\beta(p)$ is continuous,

$$
\begin{aligned}
v_\beta(0) &= \lim_{p\downarrow 0} v_\beta(p) = \beta^{-\frac{1}{2}}\lim_{p\downarrow 0}p^{-\frac{1}{2}}v(p),\\
v_\beta(0) &= \lim_{p\downarrow 0} v_\beta(-p) = \beta^{-\frac{1}{2}}\lim_{p\downarrow 0}p^{-\frac{1}{2}}v^\star(p).
\end{aligned}
$$

This implies the existence of the limits in (7.57) and the identities (7.58).

The identity (7.59) follows from (6.49) if we take into account the identities (7.52) and (7.58). □

Let

$$
\begin{aligned}
\mathfrak{N} := \Big\{B\in\mathcal{B}(\mathcal{K}):\quad & B\otimes 1\,\widetilde{v}^p(p) = \widetilde{v}^p(p)B, \quad p\in\mathcal{F}^+,\\
& B^*\otimes 1\,\widetilde{v}^p(p) = \widetilde{v}^p(p)B^*, \quad p\in\mathcal{F}^+,\\
& B\otimes 1\,\widetilde{v_{\mathrm{ir}}}^0 = \widetilde{v_{\mathrm{ir}}}^0 B\Big\}.
\end{aligned}
$$

Proposition 7.7 *$\mathfrak{N}$ is a $*$-subalgebra of $\mathcal{B}(\mathcal{K})$ containing $\mathbb{C}1_{\mathcal{K}}$. Moreover, for any $t\in\mathbb{C}$ and $B\in\mathfrak{N}$ we have $\mathrm{e}^{\mathrm{i}tK}B\mathrm{e}^{-\mathrm{i}tK}\in\mathfrak{N}$.*

Proof. It is easy to check that $\mathfrak{N}$ is an algebra and obviously $1_{\mathcal{K}}\in\mathfrak{N}$. To see that $\mathfrak{N}$ is preserved by $*$, note that the first two conditions are manifestly symmetric wrt $*$. The relation $(\widetilde{v_{\mathrm{ir}}}^0)^\star = \widetilde{v_{\mathrm{ir}}}^0$ and Proposition 4.6 imply that

$$
B\otimes 1\,\widetilde{v_{\mathrm{ir}}}^0 = \widetilde{v_{\mathrm{ir}}}^0 B \quad\Rightarrow\quad B^*\otimes 1\,\widetilde{v_{\mathrm{ir}}}^0 = \widetilde{v_{\mathrm{ir}}}^0 B^*,
$$

and so $\mathfrak{N}$ is a $*$-algebra.

Note that

$$\mathrm{e}^{\mathrm{i}tK}\otimes 1\,\widetilde{v}^p(p)\mathrm{e}^{-\mathrm{i}tK} = \mathrm{e}^{-\mathrm{i}tp}\widetilde{v}^p(p), \quad p\in\mathcal{F}^+,$$
$$\mathrm{e}^{\mathrm{i}tK}\otimes 1\,\widetilde{v_{\mathrm{ir}}}^0\mathrm{e}^{-\mathrm{i}tK} = \widetilde{v_{\mathrm{ir}}}^0.$$

This implies that $\mathfrak{N}$ is invariant wrt $\mathrm{e}^{\mathrm{i}tK}\cdot\mathrm{e}^{-\mathrm{i}tK}$. □

Proposition 7.8 $\mathrm{Ker}\Gamma_\beta^{\mathrm{I}}$ *consists of operators of the form* $\mathrm{e}^{-\beta K/2}C$ *with* $C\in\mathfrak{N}$.

Proof. Relation (7.59) can be rewritten as

$$\begin{aligned}
-\mathrm{Tr}B_1^*\Gamma_\beta^{\mathrm{I}}(B_2) \quad &= \pi\sum_{p\in\mathcal{F}^+}|\mathrm{e}^{\beta p}-1|^{-1}\mathrm{Tr}\Big(\widetilde{v}^p(p)\mathrm{e}^{\beta K/2}B_1 - \mathrm{e}^{\beta K/2}B_1\otimes 1\;\widetilde{v}^p(p)\Big)^*\\
&\qquad\times\mathrm{e}^{-\beta K}\otimes 1\Big(\widetilde{v}^p(p)\mathrm{e}^{\beta K/2}B_2 - \mathrm{e}^{\beta K/2}B_2\otimes 1\;\widetilde{v}^p(p)\Big)\\
&+\pi\sum_{p\in\mathcal{F}^+}|\mathrm{e}^{-\beta p}-1|^{-1}\mathrm{Tr}\Big(\widetilde{v}^p(p)B_2^*\mathrm{e}^{\beta K/2} - B_2^*\mathrm{e}^{\beta K/2}\otimes 1\;\widetilde{v}^p(p)\Big)^*\\
&\qquad\times\Big(\widetilde{v}^p(p)B_1^*\mathrm{e}^{\beta K/2} - B_1^*\mathrm{e}^{\beta K/2}\otimes 1\;\widetilde{v}^p(p)\Big)\mathrm{e}^{-\beta K}\\
&+\tfrac{\pi}{\beta}\mathrm{Tr}\Big(\widetilde{v_{\mathrm{ir}}}^0\mathrm{e}^{\beta K/2}B_1 - \mathrm{e}^{\beta K/2}B_1\otimes 1\;\widetilde{v_{\mathrm{ir}}}^0\Big)^*\\
&\qquad\times\mathrm{e}^{-\beta K}\otimes 1\Big(\widetilde{v_{\mathrm{ir}}}^0\mathrm{e}^{\beta K/2}B_2 - \mathrm{e}^{\beta K/2}B_2\otimes 1\;\widetilde{v_{\mathrm{ir}}}^0\Big).
\end{aligned}$$

Hence, $B\in\mathrm{Ker}\Gamma_\beta^{\mathrm{I}}$ iff

$$\begin{aligned}
&\widetilde{v}^p(p)\mathrm{e}^{\beta K/2}B - \mathrm{e}^{\beta K/2}B\otimes 1\;\widetilde{v}^p(p) = 0, \quad p\in\mathcal{F}^+,\\
&\widetilde{v}^p(p)B^*\mathrm{e}^{\beta K/2} - B^*\mathrm{e}^{\beta K/2}\otimes 1\;\widetilde{v}^p(p) = 0, \quad p\in\mathcal{F}^+,\\
&\widetilde{v}_{\mathrm{ir}}^0\mathrm{e}^{\beta K/2}B - \mathrm{e}^{\beta K/2}B\otimes 1\;\widetilde{v}_{\mathrm{ir}}^0 = 0.
\end{aligned}$$

Therefore, $B\in\mathrm{Ker}\Gamma_\beta^{\mathrm{I}}$ iff $B=\mathrm{e}^{-\beta K/2}C$ for some $C\in\mathfrak{N}$. □

Our main effective coupling assumption is:

Assumption 7.C $\mathfrak{N}=\mathbb{C}1_{\mathcal{K}}$.

Now Proposition 7.8 implies immediately

Theorem 7.9 *Assumption 7.C is satisfied iff* $\mathrm{Ker}\Gamma_\beta^{\mathrm{I}}$ *is spanned by* γ_β.

7.4 Return to equilibrium for a fixed positive temperature

In this subsection we describe conditions which ensure that for any fixed positive temperature the thermal Pauli-Fierz system has the property of return to equilibrium. The result will not be uniform in the temperature.

Theorem 7.10 *Let $\eta > 2$ and suppose that Assumptions 7.A, 7.B(η) and 7.C hold. Then for all $\beta \in]0, \infty[$ there exists $\lambda_0(\beta) > 0$ such that for $0 < |\lambda| < \lambda_0(\beta)$,*

$$\mathrm{sp_p}(L_\beta) = \{0\}, \qquad \dim 1_0(L_\beta) = 1, \qquad \mathrm{sp_{sc}}(L_\beta) = \emptyset.$$

In particular, under the above conditions the Pauli-Fierz system $(\mathfrak{M}_\beta, \tau_\beta)$ has the property of return to equilibrium.

Proof. It follows from Theorem 7.9 that $\dim \mathrm{Ker}\Gamma_\beta^{\mathrm{I}} = 1$ for $\beta \in]0, \infty[$. Hence, by Theorem 5.5, there exists $\lambda_0(\beta)$ such that for $0 < |\lambda| < \lambda_0(\beta)$ we have

$$\dim 1_\mathrm{p}(L_\beta) \leq \dim 1_0(\Gamma_\beta^{\mathrm{I}}) = 1, \qquad \mathrm{sp_{sc}}(L_\beta) = \emptyset.$$

By Theorem 7.3, $\dim 1_0(L_\beta) \geq 1$, and the result follows. $\Box$

7.5 Zero temperature Pauli-Fierz Liouvilleans

Recall that zero temperature corresponds to $\beta = \infty$. All the properties of the zero temperature Liouvillean, denoted L_∞, follow easily from the properties of the Pauli-Fierz Hamiltonian. They are described in this subsection.

Note that after the identification $\mathcal{K} \otimes \overline{\mathcal{K}} \otimes \Gamma_\mathrm{s}(\mathcal{Z} \oplus \overline{\mathcal{Z}}) \simeq \mathcal{K} \otimes \Gamma_\mathrm{s}(\mathcal{Z}) \otimes \overline{\mathcal{K} \otimes \Gamma_\mathrm{s}(\mathcal{Z})}$ the zero temperature Liouvillean becomes

$$L_\infty = H \otimes 1 - 1 \otimes \overline{H}.$$

Hence, the Level Shift Operator for L_∞, denoted Γ_∞, can be expressed in terms of the Level Shift Operator for H, denoted Γ, as follows. If Γ_∞ is expresed in terms Δ_∞ and Ξ_∞ as in Theorem 6.11, then

$$\Delta_\infty = \Gamma, \qquad \Xi_\infty = 0, \tag{7.60}$$

and so

$$\Gamma_\infty(B) = \Gamma B - B\Gamma^*.$$

The following theorem then follows immediately from Theorem 6.4:

Theorem 7.11 *Under assumptions of Theorem 6.4, there exists $\lambda_0 > 0$ such that for $0 < |\lambda| < \lambda_0$ we have*

$$\mathrm{sp_p}(L_\infty) \subset \{0\}, \qquad \dim 1_0(L_\infty) \leq 1, \qquad \mathrm{sp_{sc}}(L_\infty) = \emptyset.$$

7.6 Uniform in temperature estimate on the Level Shift Operator

In this subsection we describe conditions under which Γ_β is uniformly dissipative on the orthogonal complement to γ_β. The effective coupling assumptions 6.D, 6.E, and 7.C will play a key role.

Recall that γ_β is defined by (7.53) for $0 \le \beta < \infty$. For $\beta = \infty$ we set

$$\gamma_\infty := 1_{k_0}(K)/\sqrt{\mathrm{Tr} 1_{k_0}(K)},$$

where $k_0 := \inf \mathrm{sp}(K)$. Note that $[0, \infty] \ni \beta \mapsto \gamma_\beta \in l^2(\mathcal{K})$ is a continuous function.

Theorem 7.12 *Let $\eta > \frac{1}{2}$ and suppose that Assumption 7.B(η) holds. Then*
(1) *The function $]0, \infty] \ni \beta \mapsto \Gamma^{\mathrm{I}}_\beta \in \mathcal{B}(l^2(\mathcal{K}))$ is continuous.*
(2) *Assume in addition that 6.D, 6.E and 7.C hold. Let $\beta_0 > 0$. Then there exists $\epsilon > 0$ such that for $\beta \in [\beta_0, \infty]$*

$$\mathrm{Tr} B^* \Gamma^{\mathrm{I}}_\beta(B) \le -\epsilon(\mathrm{Tr} B^* B - |\mathrm{Tr} B\gamma_\beta|^2). \tag{7.61}$$

Proof. The continuity of $\Gamma^{\mathrm{I}}_\beta$ in $\beta \in]0, \infty]$ follows from Relation (7.59).

To prove (2), let us consider first the case $\beta = \infty$. Assumption 6.D and (7.60) imply that there exists $\epsilon(\infty) > 0$ such that

$$\Delta^{\mathrm{I}}_\infty \le -\epsilon(\infty)(1 - 1_{k_0}(K)). \tag{7.62}$$

Since $\Gamma^{\mathrm{I}}_\infty(B) = \Delta^{\mathrm{I}}_\infty B + B\Delta^{\mathrm{I}}_\infty$, using (7.62) and Assumption 6.E we obtain

$$\begin{aligned}
\mathrm{Tr} B^* \Gamma^{\mathrm{I}}_\infty(B) &\le -\epsilon(\infty)\big(\mathrm{Tr} B^*(1 - 1_{k_0}(K))B + \mathrm{Tr} B^* B(1 - 1_{k_0}(K))\big) \\
&\le -\epsilon(\infty)\big(\mathrm{Tr} B^* B - \mathrm{Tr} B^* 1_{k_0}(K) B 1_{k_0}(K)\big) \\
&= -\epsilon(\infty)(\mathrm{Tr} B^* B - |\mathrm{Tr} B\gamma_\infty|^2).
\end{aligned}$$

Consider now $\beta < \infty$. It follows from Assumption 7.C and Theorem 7.9 that for any $\beta \in]0, \infty[$, there exists $\epsilon(\beta) > 0$ such that

$$\mathrm{Tr} B^* \Gamma^{\mathrm{I}}_\beta(B) \le -\epsilon(\beta)(\mathrm{Tr} B^* B - |\mathrm{Tr} B\gamma_\beta|^2).$$

The compactness of $[\beta_0, \infty]$, the continuity of $[\beta_0, \infty] \ni \beta \mapsto \Gamma^{\mathrm{I}}_\beta$ and of $[\beta_0, \infty] \ni \beta \mapsto \gamma_\beta$ imply that one can chose $\epsilon > 0$ such that (7.61) holds. □

7.7 Uniform in temperature return to equilibrium

In this subsection we describe the main result of this paper. We describe conditions under which for $0 < |\lambda| < \lambda_0$ and $\beta \in [\beta_0, \infty[$ the Liouvillean L_β has purely absolutely continuous spectrum except for a simple eigenvalue at zero. This implies that for a small nonzero coupling constant the system $(\mathfrak{M}, \tau_\beta)$ has the property of return to equilibrium uniformly in the temperature.

One of the ingredients of our proof is the uniform estimate on the Level Shift Operator Γ_β obtained in the previous subsection. The second ingredient is an additional assumption on the regularity of the interaction, which we will formulate below.

For any $\eta \geq 0$ we introduce

Assumption 7.D(η) $\sup_{1\leq\beta\leq\infty} \|\langle s\rangle^\eta v_\beta\| < \infty$.

Proposition 7.13 *Suppose that Assumption 7.D(η) holds. Then*

(1) *Assumption 7.B(η) holds.*

(2) *For any $\beta_0 > 0$, $\sup_{\beta_0\leq\beta\leq\infty} \|\langle s\rangle^\eta v_\beta\| < \infty$.*

Proof. By Assumption 7.D(η), $\|\langle s\rangle^\eta |1-\mathrm{e}^{-r}|^{-1/2}(v, v^\star)\| < \infty$. Clearly, $\tau\big(\langle s\rangle^\eta |1-\mathrm{e}^{-r}|^{-1/2}(v, v^\star)\big)^\star = \langle s\rangle^\eta |1-\mathrm{e}^{r}|^{-1/2}(v, v^\star)$. Thus, using the boundedness of $\star$, we obtain

$$\|\langle s\rangle^\eta \big(|1-\mathrm{e}^{-r}|^{-1/2} + |1-\mathrm{e}^{r}|^{-1/2}\big)(v, v^\star)\| < \infty. \tag{7.63}$$

Using the arguments of the proof of Proposition 7.4 we see that

$$\langle s\rangle^\eta \langle r\rangle^{1/2} |r|^{-1/2} \big(|1-\mathrm{e}^{-r}|^{-1/2} + |1-\mathrm{e}^{r}|^{-1/2}\big)^{-1} \langle s\rangle^{-\eta} \tag{7.64}$$

is bounded. Now (7.63) and (7.64) imply Assumption 7.B(η), which proves (1).

For $0 < \beta_0 \leq 1$, we have

$$\sup_{\beta_0\leq\beta\leq1} \|\langle s\rangle^\eta |1-\mathrm{e}^{-\beta r}|^{-1/2} \big(|1-\mathrm{e}^{-r}|^{-1/2} + |1-\mathrm{e}^{r}|^{-1/2}\big)^{-1} \langle s\rangle^{-\eta}\| < \infty.$$

Hence (7.63) implies $\sup_{\beta_0\leq\beta\leq1} \|\langle s\rangle^\eta v_\beta\| < \infty$. This proves (2). □

Theorem 7.14 *Suppose Assumptions 7.A, 7.D(η) with $\eta > 2$, 6.D 6.E and 7.C are satisfied. Let $0 < \beta_0 < \infty$. Then there exists $\lambda_0 > 0$ such that for $0 < |\lambda| < \lambda_0$ and $\beta \in [\beta_0, \infty[$ we have*

$$\mathrm{sp_p}(L_\beta) = \{0\}, \qquad \dim 1_0(L_\beta) = 1, \qquad \mathrm{sp_{sc}}(L_\beta) = \emptyset.$$

Hence, under the same conditions, the Pauli-Fierz system $(\mathfrak{M}_\beta, \tau_\beta)$ has the property of return to equilibrium. Moreover, for $0 < |\lambda| < \lambda_0$,

$$\mathrm{sp_p}(L_\infty) \subset \{0\}, \qquad \dim 1_0(L_\infty) \leq 1, \qquad \mathrm{sp_{sc}}(L_\infty) = \emptyset.$$

Proof. By Theorem 7.2 the operator L_β is essentially self-adjoint on $\mathcal{D}(L_{\rm fr}) \cap \mathcal{D}(\pi(V_\beta) - J\pi(V_\beta)J)$ for all $\beta \in]0,\infty[$ and $\lambda \in \mathbb{R}$. By Theorem 6.3 the operator L_∞ is self-adjoint on $\mathcal{D}(L_{\rm fr}) = \mathcal{D}(L_{\rm fr}) \cap \mathcal{D}(\pi(V_\infty) - J\pi(V_\infty)J)$ for all λ.

By Theorem 7.12 there exists $\epsilon > 0$ such that for all $\beta \in [\beta_0, \infty]$,

$$\Gamma_\beta^{\rm I} \leq -\epsilon(1 - 1_0(\Gamma_\beta^{\rm I})).$$

By Assumption 7.D(η) and Proposition 7.13 (2), for all $\beta \in [\beta_0, \infty]$ and $\eta > 2$,

$$\|\langle s\rangle^\eta \big(1_{\overline{\mathcal{K}}} \check{\otimes} v_\beta - 1_{\mathcal{K}} \otimes \overline{\tau v_\beta}\big)\| \leq 2c.$$

Hence, all conditions of Theorem 5.5 are satisfied, and there exists $\lambda_0 > 0$ such that for $0 < |\lambda| < \lambda_0$ and all $\beta \in [\beta_0, \infty]$ we have

$$\dim 1^{\rm p}(L_\beta) \leq \dim \Gamma_\beta^{\rm I} = 1, \qquad {\rm sp}_{\rm sc}(L_\beta) = \emptyset.$$

By Theorem 7.3 $\dim 1_0(L_\beta) \geq 1$ for $\beta \in]0, \infty[$, and the statement follows. □

7.8 Simplified condition for return to equilibrium

In this subsection we describe conditions which are easy to verify in practice and which imply Assumption 7.D(η).

Proposition 7.15 *Suppose that n is a nonnegative integer and*

$$\|\langle r\rangle^{1/2} s^n |r|^{-1/2}(v, v^\star)\| < \infty; \tag{7.65}$$

$$\||r|^{1/2-n+j} s^j |r|^{-1/2}(v, v^\star)\| < \infty, \quad j = 0, \dots, n-1. \tag{7.66}$$

Then Assumption 7.D(n) holds.

Proof. Set $g(p) := |1 - {\rm e}^{-p}|^{-1/2}|p|^{1/2}$. Note that g is smooth and

$$|\partial_p^j g(p)| \leq c_j \langle p\rangle^{1/2-j}. \tag{7.67}$$

Now

$$\begin{aligned} s^n|1 - {\rm e}^{-\beta r}|^{-1/2}(v, v^\star) &= \beta^{-1/2} s^n g(\beta r)|r|^{-1/2}(v, v^\star) \\ &= \textstyle\sum_{j=0}^n (-{\rm i})^{n-j}\beta^{-1/2+n-j} g^{(n-j)}(\beta r) s^j |r|^{-1/2}(v, v^\star). \end{aligned}$$

To estimate the term with $j = n$ we use that for $\beta \geq 1$ we have

$$\beta^{-1/2} g(\beta r) \leq \beta^{-1/2}\langle \beta r\rangle^{1/2} \leq \langle r\rangle^{1/2}$$

To estimate the terms with $j = 0, \dots, n-1$ we use

$$\beta^{-1/2+n-j} g^{(n-j)}(\beta r) \leq h_{n-j}(\beta r)|r|^{1/2-n+j},$$

where, by (7.67), $h_{n-j}(p) = g^{(n-j)}(p)|p|^{-1/2+n-j}$ is a bounded function. □

Here are yet another set of conditions implying Assumption 7.D(n).

Proposition 7.16 *Suppose that*

$$\int_0^\infty \langle p\rangle \|\partial_p^n p^{-1/2} v(p)\|^2 \mathrm{d}p < \infty; \tag{7.68}$$

$$\int_0^\infty p^{1-2n+2j} \|\partial_p^j p^{-1/2} v(p)\|^2 \mathrm{d}p < \infty, \qquad j = 0, \dots, n-1; \tag{7.69}$$

$$\partial_p^j p^{-1/2} v(p)\Big|_{p=0} = (-1)^j \partial_p^j p^{-1/2} v^\star(p)\Big|_{p=0}, \qquad j = 0, \cdots, n-1. \tag{7.70}$$

Then (7.65) and (7.66) hold, and hence Assumption 7.D(n) holds.

Proof. Since $\mathcal{K}$ is finite dimensional, the $\star$ conjugation is a bounded linear map, and (7.68) and (7.69) hold with $v^\star$ instead of v.

We will prove (7.65). A similar argument yields (7.66). Using (7.68) and (7.70) we see that

$$\left(\langle r\rangle^{1/2} s^n |r|^{-1/2}(v, v^\star)\right)(p) = (-\mathrm{i})^n \langle p\rangle^{1/2} \begin{cases} \partial_p^n p^{-1/2} v(p), & p \geq 0, \\ \partial_p^n |p|^{-1/2} v^\star(|p|), & p \leq 0, \end{cases}$$

and

$$\begin{aligned} \|\langle r\rangle^{1/2} s^n |r|^{-1/2}(v, v^\star)\|^2 &\leq \int_{-\infty}^{\infty} \langle p\rangle \|\partial_p^n |p|^{-1/2}(v, v^\star)(p)\|^2 \mathrm{d}p \\ &= \int_0^\infty \langle p\rangle \left(\|\partial_p^n p^{-1/2} v(p)\|^2 + \|\partial_p^n p^{-1/2} v^\star(p)\|^2\right) \mathrm{d}p. \end{aligned}$$

□

7.9 Pauli-Fierz systems with several thermal reservoirs

In this subsection we prove that a generic Pauli-Fierz system with a small nonzero coupling constant has no normal invariant states. For shortness, we restrict ourselves to a result non-uniform in the temperature.

We consider the same framework as in Subsection 6.8. Moreover, we assume that the energy density of the i-th reservoir is given by

$$\rho_{\beta_i} = (\mathrm{e}^{\beta_i h_i} - 1)^{-1},$$

where $\beta_i \in]0, \infty[$. We set $\vec{\beta} = (\beta_1, \dots, \beta_N)$ and, after replacing β with $\vec{\beta}$, we adopt the same notational convention as in Subsection 7.1.

Theorem 7.17 *Let $\eta > 1$. Assume that Assumptions 7.A and 7.B(η) hold for $i = 1, \dots, N$. Suppose also that $\beta_j \neq \beta_k$ for some $j, k \in \{1, \dots, N\}$, and that Assumption 7.C holds for the jth and kth reservoir. Then there exists $\lambda_0(\vec{\beta}) > 0$ such that for $0 < |\lambda| < \lambda_0(\vec{\beta})$,*

$$\mathrm{sp}_{\mathrm{p}}(L_{\vec{\beta}}) = \emptyset, \qquad \mathrm{sp}_{\mathrm{sc}}(L_{\vec{\beta}}) = \emptyset.$$

Consequently, under the above conditions the system $(\mathfrak{M}_{\vec{\beta}}, \tau_{\vec{\beta}})$ has no normal invariant states.

Proof. The proof is very similar to the proof of Theorem 7.10. By Theorem 5.5 and the remark after it, it suffices to show that $\dim \mathrm{Ker}\Gamma^{\mathrm{I}}_{\vec{\beta}} = \{0\}$. The relation (6.51) yields

$$\mathrm{Ker}\Gamma^{\mathrm{I}}_{\vec{\beta}} = \bigcap_{i=1}^{N} \mathrm{Ker}\Gamma^{\mathrm{I}}_{i,\beta_i},$$

where Γ_{i,β_i} is the Level Shift Operator of the i-th subsystem. By Assumption 7.C, $\mathrm{Ker}\Gamma^{\mathrm{I}}_{i,\beta_i}$ is spanned by γ_{β_i} for $i = j, k$. Since $\beta_j \neq \beta_k$, $\mathrm{Ker}\Gamma^{\mathrm{I}}_{j,\beta_j} \cap \mathrm{Ker}\Gamma^{\mathrm{I}}_{k,\beta_k} = \{0\}$.$\square$

8 Examples of gluing

As we have already emphasized, the key ingredient of our method is the Jakšić-Pillet gluing condition. In this section we show that this condition is satisfied in a certain class of physically motivated models involving massless bosons. The gluing is accomplished by passing to the radial coordinates in the momentum representation.

8.1 Massless scalar bosons

In this subsection we consider the same model as in Section 3. Recall that $\mathcal{Z} = L^2(\mathbb{R}^d)$, where $\xi \in \mathbb{R}^d$ describes the momentum, and that h is the operator of multiplication by $|\xi|$. The gluing map is defined as

$$L^2(\mathbb{R}^d) \oplus \overline{L^2(\mathbb{R}^d)} \ni (f_+, \overline{f}_-) \mapsto f \in L^2(\mathbb{R}) \otimes L^2(S^{d-1}), \tag{8.71}$$

$$f(p,\omega) := \begin{cases} p^{\frac{d-1}{2}} f_+(p\omega), & p > 0, \\ (-p)^{\frac{d-1}{2}} \overline{f}_-(-p\omega), & p \leq 0. \end{cases} \tag{8.72}$$

Here, $(p,\omega) \in \mathbb{R} \times S^{d-1}$. Moreover, the conjugation in $L^2(\mathbb{R}^d)$ and $L^2(\mathbb{R}) \otimes L^2(S^{d-1})$ is the standard complex conjugation. The map (8.71) is unitary.

As in Section 3, we fix a form-factor $v : \mathbb{R}^d \mapsto \mathcal{B}(\mathcal{K})$. Recall that the corresponding Pauli-Fierz Hamiltonian is

$$\begin{aligned} H \;&:= K \otimes 1 + 1 \otimes \int |\xi| a^*(\xi) a(\xi) |\xi| \, \mathrm{d}\xi \\ &+\lambda \int \big(v(\xi) \otimes a^*(\xi) + v^*(\xi) \otimes a(\xi) \big) \mathrm{d}\xi. \end{aligned}$$

We fix the density $\mathbb{R}^d \ni \xi \mapsto \rho(\xi) \in \mathbb{R}_+$ and assume that Assumption 3.A holds. Recall that the semi-Liouvillean at density ρ is given by

$$\begin{aligned} L^{\mathrm{semi}}_{\rho} \;&:= K \otimes 1 + 1 \otimes \int \big(|\xi| a^*_{\mathrm{l}}(\xi) a_{\mathrm{l}}(\xi) - |\xi| a^*_{\mathrm{r}}(\xi) a_{\mathrm{r}}(\xi) \big) \, \mathrm{d}\xi \\ &+\lambda \int v(\xi) \otimes \big((1+\rho(\xi))^{\frac{1}{2}} a^*_{\mathrm{l}}(\xi) + \rho(\xi)^{\frac{1}{2}} a_{\mathrm{r}}(\xi) \big) \mathrm{d}\xi + \mathrm{hc}. \end{aligned}$$

If we use the glued variables and introduce

$$v_\rho(p,\omega) := \begin{cases} p^{\frac{d-1}{2}} (1+\rho(p\omega))^{\frac{1}{2}} v(p\omega), & p > 0 \\ (-p)^{\frac{d-1}{2}} \rho(-p\omega)^{\frac{1}{2}} v^*(-p\omega), & p \leq 0, \end{cases} \tag{8.73}$$

then the semi-Liouvillean can be written as

$$\begin{aligned} L_\rho^{\mathrm{semi}} &= K\otimes 1 + 1\otimes \int p a^*(p,\omega)a(p,\omega)\,\mathrm{d}p\mathrm{d}\omega \\ &+\lambda \int \big(v_\rho(p,\omega)\otimes a^*(p,\omega) + v_\rho^*(p,\omega)\otimes a(p,\omega)\big)\,\mathrm{d}p\mathrm{d}\omega. \end{aligned}$$

Now, using (8.73), it is easy to give explicit conditions on $v(\xi)$ needed for our results. For instance, Assumption 6.C$(n)_\rho$ is satisfied if

$$\int \|v_\rho(p,\omega)\|^2 \mathrm{d}p\mathrm{d}\omega < \infty, \qquad \int \|\partial_p^n v_\rho(p,\omega)\|^2 \mathrm{d}p\mathrm{d}\omega < \infty. \tag{8.74}$$

Recall that in (6.35) we introduced the antilinear map κ on $\mathcal{Z}$. In the context of scalar fields it is equal to

$$\kappa f(\xi) = \overline{f}(-\xi).$$

and satisfies $\kappa^2 = 1$ (it is an internal conjugation). Assume that

$$\rho\kappa = \kappa\rho, \quad v^\star = v.$$

In the context of scalar fields this means

$$v^*(\xi) = v(-\xi), \qquad \rho(\xi) = \rho(-\xi). \tag{8.75}$$

Then

$$v_\rho(p,\omega) = \begin{cases} p^{\frac{d-1}{2}}(1+\rho(p\omega))^{\frac12} v(p\omega), & p>0 \\ (-p)^{\frac{d-1}{2}}\rho(p\omega)^{\frac12} v(p\omega), & p\le 0. \end{cases} \tag{8.76}$$

Assume further that $\rho_\beta(\xi) := (\mathrm{e}^{\beta|\xi|}-1)^{-1}$ and set $v_\beta := v_{\rho_\beta}$. Then

$$v_\beta(p,\omega) = \left(\frac{p}{1-\mathrm{e}^{-\beta p}}\right)^{\frac12} |p|^{\frac{d}{2}-1} v(p\omega). \tag{8.77}$$

Now, if $\mathbb{R}\ni p\mapsto |p|^{\frac{d}{2}-1}v(p\,\cdot)\in\mathcal{B}(\mathcal{K},\mathcal{K}\otimes L^2(S^{d-1}))$ is n times differentiable and

$$\int \|\partial_p^j |p|^{\frac{d}{2}-1}\langle p\rangle^{1/2} v(p\omega)\|^2 \mathrm{d}p\mathrm{d}\omega < \infty, \quad j=0,n, \tag{8.78}$$

then (8.74) holds (the "gluing condition" (7.70) is automatically satisfied for $\rho=\rho_\beta$ by the differentiability $|p|^{\frac{d}{2}-1}v(p\,\cdot)$ at zero). Theorem 7.10 (the nonuniform result on return to equilibrium) applies whenever (8.74) holds with $n=3$.

The case $d=3$ and $v(\xi)\sim|\xi|^{-1/2}$ as $|\xi|\downarrow 0$, is sometimes called the "ohmic case" and is typical for the infrared regime of QED. Note that Theorem 7.10 covers the ohmic case. However, Theorem 7.14 (our uniform result) does not apply to the ohmic case.

8.2 Massless vector particles

In this section we briefly explain how Jakšić-Pillet gluing works for transversal massless vector bosons (e.g. photons).

Consider first the Hilbert space of square integrable vector fields on $\mathbb{R}^d$, that is $L^2(\mathbb{R}^d) \otimes \mathbb{C}^d \simeq L^2(\mathbb{R}^d, \mathbb{C}^d)$ and the function

$$\mathbb{R}^d \ni \xi \mapsto P_{\mathrm{tr}}(\xi) := 1 - |\xi|^{-2}|\xi)(\xi| \in \mathcal{B}(\mathbb{C}^d),$$

where $|\xi|^{-2}|\xi)(\xi|$ denotes the orthogonal projection onto $\xi/|\xi|$. We can view P_{tr} as an operator in $L^2(\mathbb{R}^d, \mathbb{C}^d)$. The Hilbert space of square integrable transversal vector fields on $\mathbb{R}^d$ is defined as $L^2_{\mathrm{tr}}(\mathbb{R}^d, \mathbb{C}^d) := \mathrm{Ran} P_{\mathrm{tr}}$.

We consider a Pauli-Fierz system with the 1-particle space $L^2_{\mathrm{tr}}(\mathbb{R}^d, \mathbb{C}^d)$ and the 1-particle energy $|\xi|$. We assume that the interaction is of the form $\mathbb{R}^d \ni \xi \mapsto v(\xi) = \big(1 - |\xi|^{-2}|\xi)(\xi|\big) v_0(\xi)$, for a certain function $\mathbb{R}^d \ni \xi \mapsto v_0(\xi) \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathbb{C}^d)$.

We can now repeat almost verbatim the constructions and statements of the previous subsection. Note that the formulas (8.73), (8.76) and (8.77) should be replaced by

$$v_\rho(p, \omega) = \big(1 - |\omega)(\omega|\big) \begin{cases} p^{\frac{d-1}{2}} (1 + \rho(p\omega))^{\frac{1}{2}} v_0(p\omega), & p > 0 \\ (-p)^{\frac{d-1}{2}} \rho(-p\omega)^{\frac{1}{2}} v_0^\star(-p\omega), & p \leq 0; \end{cases} \tag{8.79}$$

$$v_\rho(p, \omega) = \big(1 - |\omega)(\omega|\big) \begin{cases} p^{\frac{d-1}{2}} (1 + \rho(p\omega))^{\frac{1}{2}} v_0(p\omega), & p > 0 \\ (-p)^{\frac{d-1}{2}} \rho(p\omega)^{\frac{1}{2}} v_0(p\omega), & p \leq 0; \end{cases} \tag{8.80}$$

$$v_\beta(p, \omega) = \big(1 - |\omega)(\omega|\big) \left(\frac{p}{1 - \mathrm{e}^{-\beta p}} \right)^{\frac{1}{2}} |p|^{\frac{d}{2}-1} v_0(p\omega). \tag{8.81}$$

The condition (8.78) can be replaced by demanding that $\mathbb{R} \ni p \mapsto |p|^{\frac{d}{2}-1} v_0(p\,\cdot) \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes L^2(S^{d-1}))$ is n times differentiable and

$$\int \|\partial_p^j |p|^{\frac{d}{2}-1} \langle p \rangle^{1/2} v_0(p\omega)\|^2 \mathrm{d}p \mathrm{d}\omega < \infty, \quad j = 0, n. \tag{8.82}$$

References

[AH] Arai, A, Hirokawa, M.: On the existence and uniqueness of ground states of the spin-boson Hamiltonian, J. Func. Anal. **151**, 455 (1997).

[Ar] Araki, H.: Relative Hamiltonian for faithful normal states of a von Neumann algebra, Pub. R.I.M.S. Kyoto Univ. **9**, 165 (1973).

[AW] Araki, H., Woods, E.J.: Representations of the canonical commutation relations describing a nonrelativistic infinite free Bose gas, J. Math. Phys. **4**, 637 (1963).

[BFS1] Bach, V., Fröhlich, J., Sigal, I.: Quantum electrodynamics of confined nonrelativistic particles, Adv. Math. **137**, 299 (1998).

[BFS2] Bach, V., Fröhlich, J., Sigal, I.: Return to equilibrium, Journ. Math. Phys. **41**, 3985 (2000).

[BFSS] Bach, V., Fröhlich, J., Sigal, I., Soffer, A.: Positive commutators and the spectrum of Pauli-Fierz Hamiltonian of atoms and molecules, Comm. Math. Phys. **207**, 557 (1999).

[BR1] Brattelli, O., Robinson D. W.: *Operator Algebras and Quantum Statistical Mechanics, Volume 1*, Springer-Verlag, Berlin, second edition 1987.

[BR2] Brattelli, O., Robinson D. W.: *Operator Algebras and Quantum Statistical Mechanics, Volume 2*, Springer-Verlag, Berlin, second edition 1996.

[Da] Davies, E. B: Markovian master equations, Comm. Math. Phys. **39**, 91 (1974).

[DG] Dereziński, J., Gérard, C.: Asymptotic completeness in quantum field theory. Massive Pauli-Fierz Hamiltonians, Rev. Math. Phys. **11**, 383 (1999).

[DJ1] Dereziński, J., Jakšić, V.: Spectral theory of Pauli-Fierz operators, Journ. Func. Anal. **180** , 241 (2001).

[DJ2] Dereziński, J., Jakšić, V.: In preparation.

[DJP] Dereziński, J., Jakšić, V., Pillet, C.-A.: Perturbation theory of W^*-dynamics, Liouvilleans and KMS-states, to appear in Rev. Math. Phys.

[FNV] Fannes, M., Nachtergale, B., Verbeure, A.: The equilibrium states of the spin-boson model, Comm. Math. Phys. **114**, 537 (1988).

[Fr] Frigerio, A.: Stationary states of quantum dynamical semigroups, Comm. Math. Phys. **63**, 269 (1978).

[FGS] Fröhlich, J., Griesemer, M., Schlein, B.: Asymptotic completeness for Rayleigh scattering, Ann. Henri Poincaré, **3**, 107 (2002).

[Ge] Gerard, C.: On the existence of ground states for massless Pauli-Fierz Hamiltonians, Ann. Henri Poincaré **1**, 443 (2000).

[HS1] Hübner, M., Spohn, S.: Radiative decay: non-perturbative approaches, Rev. Math. Phys. **8**, 549 (1996).

[HS2] Hübner, M., Spohn, S.: Spectral properties of the spin-boson Hamiltonian, Ann. Inst. Henri Poincaré **62**, 289 (1995).

[Ja] Jadczyk, A. Z.: On some groups of automorphisms of von Neumann algebras with cyclic and separating vectors, Comm. Math. Phys. **13**, 142 (1969).

[JP1] Jakšić, V., Pillet, C.-A.: On a model for quantum friction II: Fermi's golden rule and dynamics at positive temperature, Comm. Math. Phys. **176**, 619 (1996).

[JP2] Jakšić, V., Pillet, C.-A.: On a model for quantum friction III: Ergodic properties of the spin-boson system, Comm. Math. Phys. **178**, 627 (1996).

[JP3] Jakšić, V., Pillet, C.-A.: Non-equilibrium steady states for finite quantum systems coupled to thermal reservoirs, Comm. Math. Phys. **226**, 131 (2002).

[M] Merkli, M.: Positive commutators in non-equilibrium quantum statistical mechanics, Comm. Math. Phys. **223**, 327 (2001).

[PF] Pauli W., Fierz, M.: Nuovo Cimento **15**, 167 (1938).

[OY] Okamoto, T., Yajima, K.: Complex scaling technique in non-relativistic qed, Ann. Inst. Henri Poincaré **42**, 391 (1981).

[Ru] Ruelle, D.: Natural nonequilibrium states in quantum statistical mechanics, J. Stat. Phys. **98**, 57 (2000).

[Sk] Skibsted, E.: Spectral analysis of N-body systems coupled to a bosonic system, Rev. Math. Phys. **10**, 989 (1998).

[Sp1] Spohn, H.: An algebraic condition for the approach to equilibrium of an open N-level system, Lett. Math. Phys. **2**, 33 (1977).

[Sp2] Spohn, H.: Ground states of the spin-boson Hamiltonian, Comm. Math. Phys. **123**, 277 (1989).

[Sp3] Spohn, H.: Ground state of a quantum particle coupled to a scalar Bose field, Lett. Math. Phys. **44**, 9 (1998).

[St] Stratila, S.: *Modular Theory in Operator Algebras*, Abacus Press, Turnbridge Wells 1981.