A note on the entropy production formula

Vojkan Jakšić[1] and Claude-Alain Pillet[2,3]

[1]Department of Mathematics and Statistics
McGill University
805 Sherbrooke Street West
Montreal, QC, H3A 2K6, Canada

[2]PHYMAT, Université de Toulon
B.P. 132, 83957 La Garde Cedex, France

[3]FRUMAM
CPT-CNRS Luminy, Case 907
13288 Marseille Cedex 9, France

Dedicated to Professor Huzihiro Araki on the occasion of his 70th birthday

Abstract. We give an elementary derivation of the entropy production formula of [JP1] based on Araki Perturbation Theory of KMS states. Using this derivation we show that the entropy production of any normal, stationary state is zero.

1 Introduction

Let $\mathcal{O}$ be a C^*-algebra, $E(\mathcal{O})$ the set of all states on $\mathcal{O}$ and $\omega \in E(\mathcal{O})$. We assume that there exists a reference C^*-dynamics σ_ω^t on $\mathcal{O}$ such that ω is a $(\sigma_\omega, -1)$-KMS state. In typical applications, $\mathcal{O}$ will be the algebra of observables of a quantum system made of several components (for example reservoirs). Then, $\mathcal{O}$ will have a product structure and ω could be a product of KMS-states, possibly at different temperatures. We denote by δ_ω the generator of σ_ω^t (*i.e.* $\sigma_\omega^t = \mathrm{e}^{t\delta_\omega}$) and by $\mathcal{D}(\delta_\omega)$ its domain. Let $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ be the GNS-representation of the algebra $\mathcal{O}$ associated to the state ω.

A state $\eta \in E(\mathcal{O})$ is called ω-normal if there exists a density matrix ρ_η on $\mathcal{H}_\omega$ such that, for all $A \in \mathcal{O}$, $\eta(A) = \mathrm{Tr}(\rho_\eta \pi_\omega(A))$. Let $\mathcal{N}_\omega$ be the set of all ω-normal states on $\mathcal{O}$.

For $\eta \in \mathcal{N}_\omega$, we denote by $\mathrm{Ent}(\eta|\omega)$ the relative entropy of Araki [Ar1, Ar2]. (We use the notational convention for relative entropy of [BR, Don].) If $\eta \notin \mathcal{N}_\omega$, we set $\mathrm{Ent}(\eta|\omega) = -\infty$. For unitary $U \in \mathcal{O}$ and $\eta \in E(\mathcal{O})$, we denote by η_U the state $\eta_U(A) \equiv \eta(U^*AU)$. The main result of this note is:

Theorem 1.1 *For any unitary $U \in \mathcal{O} \cap \mathcal{D}(\delta_\omega)$ and any $\eta \in E(\mathcal{O})$,*

$$\mathrm{Ent}(\eta_U|\omega) = \mathrm{Ent}(\eta|\omega) - \mathrm{i}\eta(U^*\delta_\omega(U)). \tag{1.1}$$

As we shall explain below, Theorem 1.1 is a natural generalization of the entropy production formula derived in [JP1, JP2]. The method of proof we will use in this note, however, is quite different from the one in [JP1]. We will reduce the proof of Theorem 1.1 to a fairly elementary application of some well known identities in Araki's theory of perturbation of KMS structure. The proof in [JP1], based on Araki-Connes cocycles, was technically more involved and restricted to *faithful* states $\eta \in \mathcal{N}_\omega$.

We now relate Equ. (1.1) to the entropy production formula of [JP1, JP2]. Assume that there exists a C^*-dynamics τ^t on $\mathcal{O}$ and that ω is τ-invariant. Let $V(t)$ be a time-dependent local perturbation, that is, $V(t)$ is norm-continuous, self-adjoint, $\mathcal{O}$-valued function on $\mathbb{R}$ (the time-independent case of [JP1] of course follows by setting $V(t) \equiv V$). The perturbed time evolution is the strongly continuous family of $*$-automorphisms of $\mathcal{O}$ given by the formula

$$\begin{aligned}\tau_V^t(A) &\equiv \tau^t(A) \\ &+ \sum_{n\geq 1} \mathrm{i}^n \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \, [\tau^{t_n}(V(t_n)), [\cdots, [\tau^{t_1}(V(t_1)), \tau^t(A)]]].\end{aligned}$$

In the interaction representation, τ_V^t is given by

$$\tau_V^t(A) = \Gamma_V^t \tau^t(A) \Gamma_V^{t*},$$

where $\Gamma_V^t \in \mathcal{O}$ is a family of unitaries satisfying the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_V^t = \mathrm{i}\Gamma_V^t \tau^t(V(t)), \qquad \Gamma_V^0 = \mathbf{1}.$$

Theorem 1.1 then has the following immediate corollary (see also Theorem 4.8 in [JP2]):

Corollary 1.2 *Assume that ω is τ-invariant and that $\Gamma_V^t \in \mathcal{D}(\delta_\omega)$. Then, for any $\eta \in E(\mathcal{O})$,*

$$\mathrm{Ent}(\eta \circ \tau_V^t|\omega) = \mathrm{Ent}(\eta|\omega) - \mathrm{i}\eta(\Gamma_V^t \delta_\omega(\Gamma_V^{t*})). \tag{1.2}$$

From now on we will consider the time-independent case $V(t) \equiv V$. If $V \in \mathcal{D}(\delta_\omega)$, then $\Gamma_V^t \in \mathcal{D}(\delta_\omega)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_V^t \delta_\omega(\Gamma_V^{t*}) = -\mathrm{i}\tau_V^t(\delta_\omega(V)). \tag{1.3}$$

Hence, (1.2) reduces to the entropy production formula of [JP1]:

$$\mathrm{Ent}(\eta \circ \tau_V^t|\omega) = \mathrm{Ent}(\eta|\omega) - \int_0^t \eta \circ \tau_V^s(\delta_\omega(V))\,\mathrm{d}s. \tag{1.4}$$

We emphasize that the above derivation of (1.4) allows for non-faithful η.

The entropy production of a state $\eta \in E(\mathcal{O})$ was defined in [JP1, JP2] by $\mathrm{Ep}_V(\eta) \equiv \eta(\delta_\omega(V))$, see also [OHI, O1, O2, Ru, Sp]. On physical grounds, it is natural to conjecture that if η is ω-normal and τ_V-invariant, then $\mathrm{Ep_V}(\eta) = 0$. For faithful η this was proven in [JP1]. Here, we establish this result in full generality.

Theorem 1.3 *Assume that ω is τ-invariant, that $V \in \mathcal{D}(\delta_\omega)$ and that η is τ_V-invariant and ω-normal. Then,*

$$\mathrm{Ep}_V(\eta) = 0.$$

Remark. If $\mathrm{Ent}(\eta|\omega) > -\infty$, then Theorem 1.3 is an immediate consequence of Equ. (1.4). The case $\mathrm{Ent}(\eta|\omega) = -\infty$ requires a separate and somewhat delicate argument.

The results of this note were announced in the recent review [JP2] where the interested reader may find additional information and references about entropy production and its role in non-equilibrium quantum statistical mechanics.

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2 Proof of Theorem 1.1

We assume that the reader is familiar with basic results of Tomita-Takesaki modular theory as discussed, for example, in [BR, DJP, Don, OP].

Let $\mathfrak{M}_\omega \equiv \pi_\omega(\mathcal{O})''$ be the enveloping von Neumann algebra. Since ω is $(\sigma_\omega, -1)$-KMS state, the vector Ω_ω is separating for $\mathfrak{M}_\omega$, and we denote by $\mathcal{P}$, J, Δ_ω the corresponding natural cone, modular conjugation and modular operator. We recall that $\Delta_\omega = \mathrm{e}^{L_\omega}$, where L_ω is the unique self-adjoint operator on $\mathcal{H}_\omega$ such that

$$\pi_\omega(\sigma_\omega^t(A)) = \mathrm{e}^{\mathrm{i}tL_\omega}\pi_\omega(A)\mathrm{e}^{-\mathrm{i}tL_\omega}, \qquad L_\omega\Omega_\omega = 0.$$

In particular, σ_ω^t extends naturally to a W^*-dynamics on $\mathfrak{M}_\omega$ which we again denote by σ_ω^t. In this context σ_ω^t is called modular dynamics.

Any state $\eta \in \mathcal{N}_\omega$ has a unique normal extension to $\mathfrak{M}_\omega$ which we denote by the same letter. Obviously, η is ω-normal iff η_U is ω-normal for all unitaries $U \in \mathcal{O}$ and so, in the proof of Theorem 1.1, we may restrict ourselves to ω-normal η's.

We will use the fact that if $\gamma : \mathfrak{M}_\omega \mapsto \mathfrak{M}_\omega$ is a $*$-automorphism, then

$$\mathrm{Ent}(\eta \circ \gamma | \omega \circ \gamma) = \mathrm{Ent}(\eta|\omega).$$

In particular,

$$\mathrm{Ent}(\eta_U|\omega) = \mathrm{Ent}(\eta|\omega_{U^*}).$$

Let Ψ_{U^*} be the unique vector representative of the state ω_{U^*} in the cone $\mathcal{P}$. A simple computation shows that

$$\Psi_{U^*} = \pi_\omega(U^*) J \pi_\omega(U^*) \Omega_\omega.$$

We will consider $P \equiv \pi_\omega(-\mathrm{i}U^*\delta_\omega(U))$ as a local perturbation of the modular group σ_ω^t. Let α^t be the locally perturbed W^*-dynamics,

$$\alpha^t(A) \equiv \mathrm{e}^{\mathrm{i}t(L_\omega+P)} A \mathrm{e}^{-\mathrm{i}t(L_\omega+P)} = \Theta_P^t \sigma_\omega^t(A) \Theta_P^{t*},$$

where $\mathrm{e}^{\mathrm{i}t(L_\omega+P)}\mathrm{e}^{-\mathrm{i}tL_\omega} \equiv \Theta_P^t \in \mathfrak{M}_\omega$ is a family of unitaries satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}\Theta_P^t = \mathrm{i}\Theta_P^t \sigma_\omega^t(P), \qquad \Theta_P^0 = \mathbf{1}. \tag{2.5}$$

Let ψ be the unique $(\alpha, -1)$-KMS state on $\mathfrak{M}_\omega$. By the Araki theory, $\Omega_\omega \in \mathcal{D}(\mathrm{e}^{(L_\omega+P)/2})$ and the unique vector representative of ψ in the natural cone $\mathcal{P}$ is

$$\Psi = \frac{\mathrm{e}^{(L_\omega+P)/2}\Omega_\omega}{\|\mathrm{e}^{(L_\omega+P)/2}\Omega_\omega\|}.$$

Another fundamental result of Araki's theory is the relation

$$\mathrm{Ent}(\eta|\psi) = \mathrm{Ent}(\eta|\omega) + \eta(P) - \log\|\mathrm{e}^{(L_\omega+P)/2}\Omega_\omega\|^2, \tag{2.6}$$

which holds for all ω-normal states η. (For η faithful, this relation was proven in [Ar1, Ar2], see also [BR]. Its extension to general η was obtained in [Don], see also the next section). Hence, to finish the proof it suffices to show that $\mathrm{e}^{(L_\omega+P)/2}\Omega_\omega = \Psi_{U^*}$.

We set $T^t \equiv U^*\sigma_\omega^t(U)$ and observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}T^t = \mathrm{i}T^t\sigma_\omega^t(-\mathrm{i}U^*\delta_\omega(U)), \qquad T^0 = \mathbf{1}.$$

Comparison with Equ. (2.5) immediately leads to $\pi_\omega(T^t) = \Theta_P^t$ and therefore

$$\begin{aligned} \mathrm{e}^{\mathrm{i}t(L_\omega+P)}\Omega_\omega &= \pi_\omega(T^t)\mathrm{e}^{\mathrm{i}tL_\omega}\Omega_\omega \\ &= \pi_\omega(U^*)\mathrm{e}^{\mathrm{i}tL_\omega}\pi_\omega(U)\Omega_\omega\,. \end{aligned} \tag{2.7}$$

Since the vector-valued function $z \mapsto \mathrm{e}^{\mathrm{i}z(L_\omega+P)}\Omega_\omega$ is analytic inside the strip $-1/2 < \mathrm{Im}\, z < 0$ and strongly continuous on its closure, analytic continuation of the identity (2.7) to $z = -\mathrm{i}/2$, yields

$$\begin{aligned} \mathrm{e}^{(L_\omega+P)/2}\Omega_\omega &= \pi_\omega(U^*)\Delta_\omega^{1/2}\pi_\omega(U)\Omega_\omega \\ &= \pi_\omega(U^*)J\pi_\omega(U^*)\Omega_\omega \\ &= \Psi_{U^*}, \end{aligned}$$

which is the desired relation.

3 Proof of Theorem 1.3

Let Ω_η be the vector representative of η in the natural cone $\mathcal{P}$. The standard Liouvillean associated to the dynamics τ_V^t is $L_V = L + \pi_\omega(V) - J\pi_\omega(V)J$, where L is the standard Liouvillean associated to τ^t. We recall that L and L_V are uniquely specified by

$$\pi_\omega(\tau^t(A)) = \mathrm{e}^{\mathrm{i}tL}\pi_\omega(A)\mathrm{e}^{-\mathrm{i}tL}, \qquad L\Omega_\omega = 0,$$

and

$$\pi_\omega(\tau_V^t(A)) = \mathrm{e}^{\mathrm{i}tL_V}\pi_\omega(A)\mathrm{e}^{-\mathrm{i}tL_V}, \qquad L_V\Omega_\eta = 0.$$

We denote by s_η the support of the state η and set $s'_\eta = Js_\eta J$. Obviously

$$s_\eta\Omega_\eta = s'_\eta\Omega_\eta = \Omega_\eta,$$

and since η is τ_V-invariant

$$\mathrm{e}^{\mathrm{i}tL_V}s_\eta = s_\eta\mathrm{e}^{\mathrm{i}tL_V}, \qquad \mathrm{e}^{\mathrm{i}tL_V}s'_\eta = s'_\eta\mathrm{e}^{\mathrm{i}tL_V}.$$

Let $\Delta_{\omega|\eta}$ be the relative modular operator. We recall that $\mathrm{Ker}\,\Delta_{\omega|\eta} = \mathrm{Ker}\, s'_\eta$,

$$J\Delta_{\omega|\eta}^{1/2}A\Omega_\eta = s'_\eta A^*\Omega_\omega,$$

for all $A \in \mathfrak{M}_\omega$ and that $\Delta_{\omega|\eta}$ is essentially self-adjoint on $\mathfrak{M}_\omega\Omega_\eta + (\mathbf{1} - s'_\eta)\mathcal{H}_\omega$. Hence

$$\Delta_{\omega\circ\tau_V^t|\eta\circ\tau_V^t} = \mathrm{e}^{-\mathrm{i}tL_V}\Delta_{\omega|\eta}\mathrm{e}^{\mathrm{i}tL_V},$$

and since η is τ_V-invariant,

$$\mathrm{e}^{\mathrm{i}tL_V}\,\Delta_{\omega|\eta}\mathrm{e}^{-\mathrm{i}tL_V} \;=\; \Delta_{\omega\circ\tau_V^{-t}|\eta} = \Delta_{\omega_{U^*}|\eta},$$

where $U^* \equiv \Gamma_V^t$.

As in the proof of Theorem 1.1, we set $P \equiv \pi_\omega(-\mathrm{i}U^*\delta_\omega(U))$ and denote by α the perturbation of the modular dynamics σ_ω by P. It follows that ω_{U^*} is the unique $(\alpha, -1)$-KMS state. Since also $\|\mathrm{e}^{(L_\omega+P)/2}\Omega_\omega\| = 1$, the basic perturbation formula of Araki-Donald (see Lemma 5.7 in [Don]) yields

$$s'_\eta \log \Delta_{\omega_{U^*}|\eta} = s'_\eta \log \Delta_{\omega|\eta} - s'_\eta P.$$

Hence,

$$\mathrm{e}^{\mathrm{i}tL_V}\, s'_\eta \log \Delta_{\omega|\eta}\mathrm{e}^{-\mathrm{i}tL_V} \;= s'_\eta \log \Delta_{\omega|\eta} - s'_\eta P,$$

and we conclude that for any real number $\lambda \neq 0$,

$$\mathrm{e}^{\mathrm{i}tL_V}\,\left(s'_\eta \log \Delta_{\omega|\eta} + \mathrm{i}\lambda\right)^{-1}\mathrm{e}^{-\mathrm{i}tL_V} \;= \left(s'_\eta \log \Delta_{\omega|\eta} - s'_\eta P + \mathrm{i}\lambda\right)^{-1}.$$

Since $\mathrm{e}^{-\mathrm{i}tL_V}\Omega_\eta = \Omega_\eta$, the second resolvent identity yields that for all real $\lambda \neq 0$,

$$(\Omega_\eta, (s'_\eta \log \Delta_{\omega|\eta} + \mathrm{i}\lambda)^{-1} s'_\eta P (s'_\eta \log \Delta_{\omega|\eta} - s'_\eta P + \mathrm{i}\lambda)^{-1}\Omega_\eta) = 0.$$

Since

$$\mathrm{s}-\lim_{\lambda\to\infty} \mathrm{i}\lambda\,\left(s'_\eta \log \Delta_{\omega|\eta} + \mathrm{i}\lambda\right)^{-1} = \mathbf{1},$$

and

$$\mathrm{s}-\lim_{\lambda\to\infty} \mathrm{i}\lambda\,\left(s'_\eta \log \Delta_{\omega|\eta} - s'_\eta P + \mathrm{i}\lambda\right)^{-1} = \mathbf{1},$$

we derive that

$$(\Omega_\eta, P\Omega_\eta) = (\Omega_\eta, s'_\eta P\Omega_\eta) = 0.$$

On the other hand, using Equ. (1.3), we get

$$P = \pi_\omega(-\mathrm{i}U^*\delta_\omega(U)) = -\int_0^t \pi_\omega(\tau_V^s(\delta_\omega(V)))\,\mathrm{d}s,$$

and since η is τ_V-invariant we conclude that

$$0 = (\Omega_\eta, P\Omega_\eta) = -\int_0^t \eta\circ\tau_V^s(\delta_\omega(V))\,\mathrm{d}s = -t\eta(\delta_\omega(V)),$$

for all t. This yields the statement.

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