MATH 589

Notes on the Weak law of large numbers. Classical degenerate convergence condition.

Let X_n be a sequence of independent random variables and $S_n := X_1 + \cdots + X_n$. We do *not* assume that expectations of X_n exist.

Theorem. The following statements are equivalent.

(A)

$$\lim_{n\to\infty} \frac{S_n}{n} = 0$$

in probability.

(B) The following holds:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \int_{|x|\geq n} \mathrm{d}\mu_{X_k} = 0, \tag{1}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \int_{|x|<n} x\mathrm{d}\mu_{X_k} = 0, \tag{2}$$

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} \left\{ \int_{|x|<n} x^2\mathrm{d}\mu_{X_k} - \left(\int_{|x|<n} x\mathrm{d}\mu_{X_k} \right)^2 \right\} = 0, \tag{3}$$

Proof that (B) $\Rightarrow$ (A). This direction we could have proven the last term.

Let

$$X_{nk} = X_k K_{\{|X_k|<n\}}, \qquad k = 1, \cdots, n, \tag{4}$$

be truncated random variables. Note that $X_{nk}(\omega) = X_k(\omega)$ if $|X_k(\omega)| < n$, otherwise $X_{nk}(\omega) = 0$. Let

$$S_{nn} := \sum_{k=1}^{n} X_{nk}. \tag{5}$$

Now,

$$\begin{aligned} P\left(\frac{S_n}{n} \neq \frac{S_{nn}}{n}\right) &\leq P\left(\bigcup_{k=1}^{n} \{X_k \neq X_{nk}\}\right) \\ &\leq \sum_{k=1}^{n} P(X_k \neq X_{nk}) \\ &= \sum_{k=1}^{n} \int_{|x|\geq n} \mathrm{d}\mu_{X_k} \to 0 \ \text{ as } n \to \infty. \end{aligned}$$

Since
$$\left|P\left(\left|\frac{S_n}{n}\right| \geq \epsilon\right) - P\left(\left|\frac{S_{nn}}{n}\right| \geq \epsilon\right)\right| \leq P\left(\frac{S_n}{n} \neq \frac{S_{nn}}{n}\right)$$
we see that (1) implies that
$$\lim_{n\to\infty} \frac{S_n}{n} = 0 \qquad \text{in probability}$$
if and only if
$$\lim_{n\to\infty} \frac{S_{nn}}{n} = 0 \qquad \text{in probability.}$$
By (2),
$$\mathbb{E}(S_{nn}/n) = \frac{1}{n}\sum_{k=1}^{n} \int_{|x|<n} x \mathrm{d}\mu_{X_k} \to 0 \text{ as } n \to \infty,$$
and to prove (A) it suffices to show that
$$\lim_{n\to\infty} \frac{S_{nn} - \mathbb{E}(S_{nn})}{n} = 0 \qquad \text{in probability.} \tag{6}$$
Note that
$$\mathrm{Var}\left(\frac{S_{nn}}{n}\right) = \frac{1}{n^2}\sum_{k=1}^{n} \mathbb{E}((X_{nk} - \mathbb{E}(X_{nk}))^2) = \frac{1}{n^2}\sum_{k=1}^{n}(\mathbb{E}(X_{nk}^2) - (\mathbb{E}(X_{nk}))^2)$$
$$= \lim_{n\to\infty} \frac{1}{n^2}\sum_{k=1}^{n}\left\{\int_{|x|<n} x^2 \mathrm{d}\mu_{X_k} - \left(\int_{|x|<n} x \mathrm{d}\mu_{X_k}\right)^2\right\}.$$
Hence, by (3),
$$\lim_{n\to\infty} \mathrm{Var}\left(\frac{S_{nn}}{n}\right) = 0,$$
and Chebyshev's inequality yields that for $\epsilon > 0$,
$$P\left(\left|\frac{S_{nn} - \mathbb{E}(S_{nn})}{n}\right| \geq \epsilon\right) \leq \frac{1}{\epsilon^2}\mathrm{Var}\left(\frac{S_{nn}}{n}\right) \to 0 \text{ as } n \to \infty.$$
This yields (6) and the statement follows.

Remark. Note how the conditions (1), (2), (3) appeared naturaly in the above proof.

Proof that (A) $\Rightarrow$ (B). This direction is more subtle. We will make use of the weak symmetrization lemma (Theorem 13-5 A) and of the truncation inequalities.

Let ψ_k be the c.f. of X_k and φ_n the c.f. of S_n/n. Obviosuly,

$$\varphi_n(t) = \prod_{k=1}^{n} \psi_k(t/n). \tag{7}$$

The characteristic function of the zero random variable is the constant 1, and by the assumption,

$$\lim_{n\to\infty} \varphi_n(t) = 1,$$

uniformly on bounded intervals. (Recall that $S_n/n \to 0$ in probability iff $S_n/n \to 0$ in law.) Let $c > 1$ be given. Then, for n large enough and all $|t| \le c$, $|\varphi_n(t) - 1| < 1/2$. This and (7) imply that for n large enough, $\psi_k(t/n) \neq 0$ for $|t| \le c$.

Since for any random variable Y, $m_{Y/n} = m_Y/n$ (m_Z denotes a median of Z), by Theorem 13-5 A and the truncation inequality,

$$\begin{aligned}
\frac{1}{2}\sum_{k=1}^{n} P\left(\left|\frac{X_k - m_{X_k}}{n}\right| \ge c^{-1}\right) &\le \sum_{k=1}^{n} P\left(\frac{|X_k^s|}{n} \ge c^{-1}\right) \\
&= \sum_{k=1}^{n} \int_{|x|\ge c^{-1}} \mathrm{d}\mu_{X_k^s/n} \\
&\le -\frac{7}{c}\sum_{k=1}^{n} \int_0^c \ln|\psi_k(t/n)|^2 \mathrm{d}t \\
&= -\frac{7}{c}\int_0^c \ln|\varphi_n(t)|^2 \mathrm{d}t.
\end{aligned}$$

Since $\lim_{n\to\infty} \ln|\varphi_n(t)|^2 = \ln 1 = 0$ uniformly on $[0, c]$, we derive

$$\lim_{n\to\infty} \sum_{k=1}^{n} P\left(\left|\frac{X_k - m_{X_k}}{n}\right| \ge c^{-1}\right) = 0. \tag{8}$$

Since

$$\frac{X_n}{n} = \frac{S_n}{n} - \frac{n-1}{n}\frac{S_{n-1}}{n-1},$$

we have that $\lim_{n\to\infty} X_n/n = 0$ in probablity. This implies that (an easy exercise)

$$\lim_{n\to\infty} \frac{m_{X_n}}{n} = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \max_{1\le k\le n} \frac{m_{X_k}}{n} = 0.$$

It follows that for n large enough and all $k \le n$, $c^{-1} + |m_{X_k}|/n < 1$ (recall that $c > 1$) and

$$P\left(\left|\frac{X_k - m_{X_k}}{n}\right| \ge c^{-1}\right) \ge \int_{|x|\ge n(c^{-1}+|m_{X_k}|/n)} \mathrm{d}\mu_{X_k} \ge \int_{|x|\ge n} \mathrm{d}\mu_{X_k}.$$

Hence, (8) yields that (1) in (B) holds. We now use again the truncation (4). Since $S_n/n \to 0$ in probability (this is our assumption), (1) implies that $S_{nn}/n \to 0$ in probability (by the same argument as in the direction (B) $\Rightarrow$ (A)). Let ψ_{nk} be the characteristic function of X_{nk} and φ_{nn} the characteristic function on S_{nn}/n. We have that

$$\varphi_{nn}(t) = \prod_{k=1}^{n} \psi_{nk}(t/n),$$

and that $\varphi_{nn}(t) \to 1$ uniformly on bounded intervals. In particular, for n large enough, $\varphi_{nn}(1) \neq 0$ (and hence $\psi_{nk}(1/n) \neq 0$ for $k \leq n$). Then

$$\begin{aligned}
2\sum_{k=1}^{n} \mathrm{Var}(X_{nk}/n) &= \sum_{k=1}^{n} \mathrm{Var}(X_{nk}^s/n) \\
&= \sum_{k=1}^{n} \int_{|x|<1} x^2 \mathrm{d}\mu_{X_{nk}^s/n} \\
&\leq \sum_{k=1}^{n} -3\ln|\psi_{nk}(1/n)|^2 \\
&= -3\ln|\varphi_{nn}(1)|^2.
\end{aligned}$$

(We have used the truncation inequality.) Since $\ln|\varphi_{nn}(1)|^2 \to \ln 1 = 0$, we have that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \mathrm{Var}(X_{nk}/n) = 0. \tag{9}$$

The relation (9) is equivalent to (3) in (B).

In remains to show that (2) in (B) holds. A simple application of Chebyshev's inequality and (9) yield that

$$\lim_{n\to\infty} \frac{S_{nn} - \mathbb{E}(S_{nn})}{n} = 0 \qquad \text{in probability.}$$

Since $S_{nn}/n \to 0$ in probability, we must have

$$\lim_{n\to\infty} \mathbb{E}(S_{nn})/n = 0.$$

This, however, is the condition (2) in (B).