MATH 589

Notes on the truncation inequalities

Let X be a random variable, μ_X its distribution and φ_X its characteristics function. Let $t > 0$ be given. Then

(A)

$$\int_{|x|<1/t} x^2 \mathrm{d}\mu_X \leq \frac{3}{t^2}(1 - \mathrm{Re}\varphi_X(t)).$$

(B)

$$\int_{|x|\geq 1/t} \mathrm{d}\mu_X \leq \frac{7}{t}\int_0^t (1 - \mathrm{Re}\varphi_X(s))\mathrm{d}s.$$

(C) If $\mathrm{Re}\varphi_X(t) > 0$, then in (A) one can replace $(1 - \mathrm{Re}\varphi_X(t))$ with $-\ln \varphi_X(t)$. Similarly, if $\mathrm{Re}\varphi_X(s) > 0$ on $[0, t]$, then in (B) one can replace $1 - \mathrm{Re}\varphi_X(s)$ with $-\ln \varphi_X(s)$.

Note that since $\mathrm{Re}\varphi_X(0) = 1$ and $\varphi_X(t)$ is continuous, (C) can be always achived by taking t sufficently small.

Proof of (A). Note first that

$$1 - \mathrm{Re}\varphi_X(t) = \int_{\mathbb{R}} (1 - \cos(tx))\mathrm{d}\mu_X \geq \int_{|x|<1/t} (1 - \cos(tx)\mathrm{d}\mu_X.$$

The Taylor expansion yields

$$1 - \cos(tx) = \frac{t^2x^2}{2} - \frac{t^4x^4}{24} + R(xt),$$

where the remainder $R(xt)$ is positive for $|xt| < 1$. Hence,

$$1 - \mathrm{Re}\varphi_X(t) \geq \int_{|x|<1/t} \frac{t^2x^2}{2}\left(1 - \frac{t^2x^2}{12}\right) \geq \frac{11t^2}{24}\int_{|x|<1/t} x^2 \mathrm{d}\mu_X.$$

This yields (A).

Proof of (B).

$$\begin{aligned}\frac{1}{t}\int_0^t (1-\mathrm{Re}\varphi_X(s))\mathrm{d}s &= \frac{1}{t}\int_0^t \int_{\mathbb{R}} (1-\cos(sx))\mathrm{d}\mu_X \mathrm{d}s\\ &= \frac{1}{t}\int_{\mathbb{R}}\int_0^t (1-\cos(sx))\mathrm{d}s\mathrm{d}\mu_X\\ &= \int_{\mathbb{R}}\left(1-\frac{\sin(tx)}{tx}\right)\mathrm{d}\mu_X\\ &\geq \int_{|x|\geq 1/t}\left(1-\frac{\sin(tx)}{tx}\right)\mathrm{d}\mu_X\\ &\geq (1-\sin 1)\int_{|x|\geq 1/t}\mathrm{d}\mu_X.\end{aligned}$$

We used that for $|a|\geq 1$, $(\sin a)/a \leq \sin 1$. Since $1-\sin 1\geq 1/7$, (B) follows.

Proof of (C). (C) follows from (A) and (B) and observation that $\ln a \leq a-1$ for $a>0$.