MATH 589

Notes on the Lindberg-Feller theorem

Let X_n be a sequence of independent random variables with finite variances. Set $\mathrm{Var}(X_n) = \sigma_n^2$ and

$$s_n = \left(\sum_{k=1}^{n} \sigma_k^2 \right)^{1/2}$$

Without loss of generality we may assume that $\mathbb{E}(X_n) = 0$. To avoid discussion of trivial cases, we also assume $s_1 \neq 0$.

Theorem. The following statements are equivalent:
(A) For all $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{k=1}^{n} \int_{|x|\geq \epsilon s_n} x^2 \mathrm{d}\mu_{X_k} = 0. \tag{1}$$

(B) $\lim_{n\to\infty} \max_{1\leq k\leq n} \sigma_k^2/s_n^2 = 0$ and the random variables

$$\frac{1}{s_n} \sum_{k=1}^{n} X_k$$

converge in law to a $\mathcal{N}(0,1)$ random variable.

Proof that (A) $\Rightarrow$ (B) (Lindberg's direction). Fix $\epsilon > 0$. Then

$$\max_{1\leq k\leq n} \frac{\sigma_k^2}{s_n^2} \leq \frac{1}{s_n^2} \max_{1\leq k\leq n} \int_{|x|<\epsilon s_n} x^2 \mathrm{d}\mu_{X_k} + \frac{1}{s_n^2} \max_{1\leq k\leq n} \int_{|x|\geq \epsilon s_n} x^2 \mathrm{d}\mu_{X_k}$$

$$\leq \epsilon^2 + \frac{1}{s_n^2} \sum_{k=1}^{n} \int_{|x|\geq \epsilon s_n} |x|^2 \mathrm{d}\mu_{X_k}.$$

Hence,

$$\limsup_{n\to\infty} \max_{1\leq k\leq n} \sigma_k^2/s_n^2 \leq \epsilon^2.$$

Since ϵ is arbitrary, $\lim_{n\to\infty} \max_{1\leq k\leq n} \sigma_k^2/s_n^2 = 0$.

We now turn to the main statement of the theorem. It is convenient to introduce the random variables

$$X_{nk} := X_k/s_n, \qquad k = 1, \cdots, n. \tag{2}$$

The Lindberg condition (1) can be rewriten as

$$\lim_{n\to\infty}\sum_{k=1}^{n}\int_{|x|\geq\epsilon} x^2 \mathrm{d}\mu_{X_{nk}} = 0, \tag{3}$$

and we need to show that (3) implies that

$$\sum_{k=1}^{n} X_{nk}$$

converges in law to a $\mathcal{N}(0,1)$ random variable.

Let $\varphi_{nk}(t)$ be the characteristic function of X_{nk} and

$$\varphi_n(t) = \prod_{k=1}^{n}\varphi_{nk}(t). \tag{4}$$

To prove the statement it suffices to show that for all t,

$$\lim_{n\to\infty}\varphi_n(t) = \mathrm{e}^{-t^2/2}. \tag{5}$$

In the proof we will use the following elementary estimates: For all real a,

$$|\mathrm{e}^{\mathrm{i}a} - 1| \leq |a|, \qquad |\mathrm{e}^{\mathrm{i}a} - 1 - \mathrm{i}a| \leq a^2/2,$$
$$|\mathrm{e}^{\mathrm{i}a} - 1 - \mathrm{i}a + a^2/2| \leq |a|^3/6.$$

We also recall that the principal branch of the complex logarithm is defined by

$$\mathrm{Ln}z = \ln|z| + \mathrm{i}\mathrm{Arg}(z),$$

where $-\pi < \mathrm{Arg}(z) \leq \pi$. $\mathrm{Ln}z$ is an analytic function in the region $\mathbb{C}\backslash\{\mathrm{Re}z \leq 0, \mathrm{Im}z = 0\}$, and $\exp(\mathrm{Ln}z) = z$. Recall that for $|z-1| < 1$,

$$\mathrm{Ln}z = \mathrm{Ln}(1 + (z-1)) = \sum_{j=1}^{\infty}(-1)^{j-1}\frac{(z-1)^j}{j}.$$

We now turn to the proof of (5). Let t be given. Then for any $\epsilon > 0$,

$$\begin{aligned}
|\varphi_{nk}(t) - 1| &\leq \left|\int_{\mathbb{R}}(\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx)\mathrm{d}\mu_{X_{nk}}\right| \\
&\leq \frac{t^2}{2}\int_{|x|<\epsilon} x^2\mathrm{d}\mu_{X_{nk}} + \frac{t^2}{2}\int_{|x|\geq\epsilon} x^2\mathrm{d}\mu_{X_{nk}} \\
&\leq \frac{t^2\epsilon^2}{2} + \sigma_k^2/s_n^2.
\end{aligned}$$

Hence,

$$\max_{1\le k\le n}|\varphi_{nk}(t)-1|\le\frac{t^2\epsilon^2}{2}+\max_{1\le k\le n}\sigma_k^2/s_n^2.$$

and it follows that

$$\lim_{n\to\infty}\max_{1\le k\le n}|\varphi_{nk}(t)-1|=0. \tag{6}$$

In particular, we have that for n large enough and all $1\le k\le n$, $|\varphi_{nk}(t)-1|<1/2$. Hence, for n large enough,

$$\sum_{k=1}^{n}\mathrm{Ln}\varphi_{nk}(t)=\sum_{k=1}^{n}(\varphi_{nk}(t)-1)+\sum_{k=1}^{n}R_{nk}(t), \tag{7}$$

where

$$R_{nk}(t)=\sum_{j=2}^{\infty}\frac{(-1)^{j-1}}{j}(\varphi_{nk}(t)-1)^j. \tag{8}$$

We estimate

$$\begin{aligned}
|R_{nk}(t)| &\le \frac{1}{2}\sum_{j=2}^{\infty}|\varphi_{nk}(t)-1|^j\\
&= \frac{1}{2}\frac{|\varphi_{nk}(t)-1|^2}{1-|\varphi_{nk}(t)-1|}\\
&\le |\varphi_{nk}(t)-1|^2\\
&= |\varphi_{nk}(t)-1|\left|\int_{\mathbb{R}}(\mathrm{e}^{\mathrm{i}tx}-1)\mathrm{d}\mu_{X_{nk}}\right|,
\end{aligned}$$

and so

$$|R_{nk}(t)|\le\frac{t^2}{2}|\varphi_{nk}(t)-1|\int_{\mathbb{R}}x^2\mathrm{d}\mu_{X_{nk}}. \tag{9}$$

Let

$$R_n(t):=\sum_{k=1}^{n}R_{nk}(t). \tag{10}$$

Since

$$\sum_{k=1}^{n}\int_{\mathbb{R}}x^2\mathrm{d}\mu_{X_{nk}}=1, \tag{11}$$

the estimate (9) yields

$$|R_n(t)|\le\sum_{k=1}^{n}|R_{nk}(t)|\le\frac{t^2}{2}\max_{1\le k\le n}|\varphi_{nk}(t)-1|, \tag{12}$$

and (6) implies that

$$\lim_{n\to\infty} R_n(t) = 0. \tag{13}$$

We now go back to (7) and show that

$$\lim_{n\to\infty} \sum_{k=1}^{n} (\varphi_{nk}(t) - 1) = -t^2/2. \tag{14}$$

Let

$$\rho_n(t) := \sum_{k=1}^{n} (\varphi_{nk}(t) - 1) + t^2/2.$$

Using (11), we rewrite

$$\rho_n(t) = \sum_{k=1}^{n} \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx + t^2x^2/2) \mathrm{d}\mu_{X_{nk}}.$$

Hence, for any $\epsilon > 0$,

$$\begin{aligned} |\rho_n(t)| \le & \sum_{k=1}^{n} \left| \int_{|x|<\epsilon} (\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx + t^2x^2/2) \mathrm{d}\mu_{X_{nk}} \right| \\ & + \sum_{k=1}^{n} \left| \int_{|x|\ge\epsilon} (\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx) \mathrm{d}\mu_{X_{nk}} \right| \\ & + \frac{t^2}{2} \sum_{k=1}^{n} \int_{|x|\ge\epsilon} x^2 \mathrm{d}\mu_{X_{nk}}. \end{aligned} \tag{15}$$

Note that

$$\begin{aligned} \sum_{k=1}^{n} \left| \int_{|x|<\epsilon} (\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx + t^2x^2/2) \mathrm{d}\mu_{X_{nk}} \right| & \le \frac{|t|^3}{6} \sum_{k=1}^{n} \int_{|x|<\epsilon} |x|^3 \mathrm{d}\mu_{X_{nk}} \\ & \le \frac{|t|^3\epsilon}{6} \sum_{k=1}^{n} \int_{\mathbb{R}} x^2 \mathrm{d}\mu_{X_{nk}} \\ & \le \frac{|t|^3\epsilon}{6}. \end{aligned} \tag{16}$$

Also,

$$\sum_{k=1}^{n} \left| \int_{|x|\ge\epsilon} (\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx) \mathrm{d}\mu_{X_{nk}} \right| \le \frac{t^2}{2} \sum_{k=1}^{n} \int_{|x|\ge\epsilon} x^2 \mathrm{d}\mu_{X_{nk}}. \tag{17}$$

Combining (16) and (17) with (15) we get that for all $\epsilon > 0$,

$$|\rho_n(t)| \le \frac{|t|^3\epsilon}{6} + t^2 \sum_{k=1}^{n} \int_{|x|\ge\epsilon} x^2 \mathrm{d}\mu_{X_{nk}}.$$

The Lindberg condition (recall (3)) implies that $\rho_n(t) \to 0$ and (14) follows.

Combining (13) and (14) with (7), we get that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \mathrm{Ln}\varphi_{nk}(t) = -t^2/2.$$

Hence, for all t,

$$\lim_{n\to\infty} \varphi_n(t) = \lim_{n\to\infty} \mathrm{e}^{\sum_{k=1}^{n} \mathrm{Ln}\varphi_{nk}(t)} = \mathrm{e}^{-t^2/2},$$

and we are done.

Exercise: Theorem 12-5C yields that the convergence $\varphi_n(t) \to \exp(-t^2/2)$ is uniform on bounded intervals. Prove this fact directly by slightly modifying the above proof.

Proof that (B) $\Rightarrow$ (A) (Feller's direction). We will use again the notation (2) and (4). We know that

$$\lim_{n\to\infty} \max_{1\le k\le n} \frac{\sigma_k^2}{s_n^2} = 0, \tag{18}$$

and that

$$\lim_{n\to\infty} \varphi_n(t) = \mathrm{e}^{-t^2/2}.$$

We want to show that the Lindberg condition holds, namely that for all $\epsilon > 0$,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \int_{|x|\ge\epsilon} x^2 \mathrm{d}\mu_{X_{nk}}(t) = 0.$$

In what follows we fix $t \ne 0$. Since

$$\begin{aligned} |\varphi_{nk}(t) - 1| &= \left| \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}x) \mathrm{d}\mu_{X_{nk}} \right| \\ &\le \frac{t^2}{2} \int_{\mathbb{R}} x^2 \mathrm{d}\mu_{X_{nk}} = \frac{t^2}{2} \frac{\sigma_k^2}{s_n^2}, \end{aligned}$$

(18) implies that

$$\lim_{n\to\infty} \max_{1\le k\le n} |\varphi_{nk}(t) - 1| = 0.$$

In particular, starting with n large enough, $\max_{1\le k\le n}|\varphi_{nk}(t)-1|<1/2$, $\mathrm{Ln}\varphi_{nk}(t)$ is well-defined and

$$\sum_{k=1}^{n}\mathrm{Ln}\varphi_{nk}(t)=\sum_{k=1}^{n}(\varphi_{nk}(t)-1)+R_n(t) \tag{19}$$

where $R_n(t)$ and $R_{nk}(t)$ are defined by (8) and (10). The estimate (12) holds again (with the same proof) and so $\lim_{n\to\infty}R_n(t)=0$. Since

$$\varphi_n(t)=\mathrm{e}^{\sum_{k=1}^{n}\mathrm{Ln}\varphi_{nk}(t)}\to\mathrm{e}^{-t^2/2}$$

(19) yields

$$\lim_{n\to\infty}\mathrm{Re}\sum_{k=1}^{n}(\varphi_{nk}(t)-1)=-t^2/2.$$

Using (11) we can rewrite the above relation as

$$\lim_{n\to\infty}\mathrm{Re}\sum_{k=1}^{n}\int_{\mathbb{R}}(\mathrm{e}^{\mathrm{i}tx}-1-\mathrm{i}tx+t^2x^2/2)\mathrm{d}\mu_{X_{nk}}=0. \tag{20}$$

Since

$$\mathrm{e}^{\mathrm{i}tx}-1-\mathrm{i}tx=-t^2x^2\int_0^1(1-s)\mathrm{e}^{\mathrm{i}stx}\mathrm{d}s,$$

(20) is equivalent to

$$\lim_{n\to\infty}\mathrm{Re}\sum_{k=1}^{n}\int_{\mathbb{R}}x^2\left[\int_0^1(1-s)(\mathrm{e}^{\mathrm{i}stx}-1)\mathrm{d}s\right]\mathrm{d}\mu_{X_{nk}}=0. \tag{21}$$

Taking the real part we get

$$\lim_{n\to\infty}\sum_{k=1}^{n}\int_{\mathbb{R}}x^2\left[\int_0^1(1-s)(1-\cos(stx))\mathrm{d}s\right]\mathrm{d}\mu_{X_{nk}}=0. \tag{22}$$

Consider now the function

$$h(x):=\int_0^1(1-s)(1-\cos(stx))\mathrm{d}s.$$

(Recall that $t\neq 0$ is fixed). The function $h(x)$ is continuous, strictly positive for $x\neq 0$, and

$$\lim_{|x|\to\infty}h(x)=1/2.$$

Hence, for any $\epsilon>0$,

$$c_\epsilon:=\inf_{|x|\ge\epsilon}h(x)>0.$$

It follows that

$$\int_{\mathbb{R}} x^2 h(x)\mathrm{d}\mu_{X_{nk}} \geq \int_{|x|\geq\epsilon} x^2 h(x)\mathrm{d}\mu_{X_{nk}} \geq c_\epsilon \int_{|x|\geq\epsilon} x^2 \mathrm{d}\mu_{X_{nk}},$$

and the relation (22) implies

$$\lim_{n\to\infty} \sum_{k=1}^{n} \int_{|x|\geq\epsilon} x^2 \mathrm{d}\mu_{X_{nk}} = 0.$$