MATH 589

Notes on infinitely divisible random variables with finite variance

We denote by $\mathcal{F}$ the collection of all infinitely divisible random variables with finite variance. Obviously, the normal random variables belong to $\mathcal{F}$. Another important class of elements of $\mathcal{F}$ are the so called Poisson random variables. Let a and $b \neq 0$ be real numbers and $\lambda > 0$. A random variable X has the Poisson $\mathcal{P}(a, b, \lambda)$ distribution if

$$P(X = a + kb) = \mathrm{e}^{-\lambda}\frac{\lambda^k}{k!}, \qquad k = 0, 1, \cdots$$

Note that

$$\begin{aligned}\varphi_X(t) = \mathbb{E}(\mathrm{e}^{\mathrm{i}tX}) &= \sum_{k=0}^{\infty} \mathrm{e}^{\mathrm{i}t(a+kb)} P(X = a + kb) \\ &= \sum_{k=0}^{\infty} \mathrm{e}^{\mathrm{i}t(a+kb)} \mathrm{e}^{-\lambda}\frac{\lambda^k}{k!} \\ &= \exp\left\{\mathrm{i}ta + \lambda(\mathrm{e}^{\mathrm{i}tb} - 1)\right\}.\end{aligned} \tag{1}$$

If $a = 0$ and $b = 1$ (i.e., X is the "standard" Poisson), we write $\mathcal{P}(\lambda)$ for $\mathcal{P}(0, 1, \lambda)$.

The following simple property of the family $\mathcal{F}$ will be important.

Theorem 1. Let X_n be a sequence of elements of $\mathcal{F}$. Assume that $\sup_n \mathrm{Var}(X_n) \leq C$ and that X_n converges in law to a random variable X. Then $X \in \mathcal{F}$ and $\lim_{n\to\infty} \mathbb{E}(X_n) = \mathbb{E}(X)$.

Proof. X is infinitely divisible by Theorem 14-2 D proven in class.

Let $m_n = \mathbb{E}(X_n)$. Assume that $\sup_n |m_n| = \infty$. Let $a < b$ be continuity points of F_X such that $F_X(b) > F_X(a)$. By the first Helly theorem,

$$\int_a^b x \mathrm{d}\mu_{X_n} \to \int_a^b x \mathrm{d}\mu_X, \qquad \int_a^b x^2 \mathrm{d}\mu_{X_n} \to \int_a^b x^2 \mathrm{d}\mu_X,$$

$$\int_a^b \mathrm{d}\mu_{X_n} \to F_X(b) - F_X(a) > 0,$$

and the identity

$$\int_a^b (x - m_n)^2 \mathrm{d}\mu_{X_n} = \int_a^b x^2 \mathrm{d}\mu_{X_n} - 2m_n \int_a^b x \mathrm{d}\mu_{X_n} + m_n^2 \int_a^b \mathrm{d}\mu_{X_n}$$

yields that $\sup_n \int_a^b (x-m_n)^2 \mathrm{d}\mu_{X_n} = \infty$. This contradicts the assumption $\sup_n \mathrm{Var}(X_n) < \infty$. Hence, we must have that $\sup_n |m_n| < \infty$. Since

$$\mathrm{Var}(X_n) = \mathbb{E}(X_n^2) - m_n^2,$$

we have that

$$C' = \sup_n \mathbb{E}(X_n^2) < \infty. \tag{2}$$

By the third Helly theorem, for any $\epsilon > 0$

$$C' \geq \lim_{n\to\infty} \int_{\mathbb{R}} x^2 \mathrm{e}^{-\epsilon x^2} \mathrm{d}\mu_{X_n} = \int_{\mathbb{R}} x^2 \mathrm{e}^{-\epsilon x^2} \mathrm{d}\mu_X$$

and so, by the Monotone convergence theorem, $\mathbb{E}(X^2) \leq C'$. Hence, $\mathrm{Var}(X) < \infty$.

The problem #8 on the Assignment 4 from the last term and the relation (2) yield that $m_n \to \mathbb{E}(X)$.□

The family $\mathcal{F}$ is characterized by the next two theorems.

Theorem 2. If $X \in \mathcal{F}$, then there exists a unique real number m and a unique bounded distribution function K such that

$$\varphi_X(t) = \exp\left\{ \mathrm{i}mt + \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx) x^{-2} \mathrm{d}K(x) \right\}. \tag{3}$$

The integrand is defined by continuty at $x = 0$ and equals $-t^2/2$. Moreover, $m = \mathbb{E}(X)$ and $K(\infty) = \mathrm{Var}(X)$.

Proof. Since X is infinitely divisible, for each n there are i.i.d. random variables X_{nk}, $1 \leq k \leq n$, such that

$$X = \sum_{k=1}^n X_{nk}$$

(in the distributional sense). Obviously, $m = \mathbb{E}(X) = n\mathbb{E}(X_{n1})$ and $\mathrm{Var}(X) = n\mathrm{Var}(X_{n1})$. Let φ_n and μ_n be the characteristic function and the distribution of $X_{n1} - \mathbb{E}(X_{n1})$. Then, by Theorem 14-3 A (proven in class),

$$\log \varphi_X(t) = \mathrm{i}m + \lim_{n\to\infty} n(\varphi_n(t) - 1) = \mathrm{i}m + \lim_{n\to\infty} n \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx) \mathrm{d}\mu_n.$$

Let

$$K_n(x) = n \int_{-\infty}^x y^2 \mathrm{d}\mu_n(y).$$

Since $K_n(\infty) = n\mathrm{Var}(X_{n1}) = \mathrm{Var}(X)$, K_n is a bounded distribution function and

$$\log \varphi_X(t) = \mathrm{i}m + \lim_{n\to\infty} \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx)x^{-2}\mathrm{d}K_n(x).$$

Since K_n is a uniformly bounded sequence of distributions functions, by Helly compactness theorem there is a subsequence K_{n_j} which converges weakly to a bounded distribution function K. Since

$$\lim_{|x|\to\infty} (\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx)x^{-2} = 0,$$

by the third Helly theorem

$$\begin{aligned}\log \varphi_X(t) &= \mathrm{i}m + \lim_{j\to\infty} \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx))x^{-2}\mathrm{d}K_{n_j}(x)\\ &= \mathrm{i}m + \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx)x^{-2}\mathrm{d}K(x).\end{aligned}$$

This establishes the representation (3) and shows that $m = \mathbb{E}(X)$.

Since for fixed x the function

$$\mathbb{R} \ni t \mapsto (\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx)x^{-2}$$

is twice continuously differentiable with uniformly bounded derivatives in (t, x), an application of the Lebesgue dominated convergence theorem yields that $\varphi_X(t)$ is twice continously differentiable and that

$$\varphi_X'(0) = \mathrm{i}m, \qquad -\varphi_X''(0) = m^2 + \int_{\mathbb{R}} \mathrm{d}K(x).$$

(Justify this step). Hence, $\mathrm{Var}(X) = K(\infty)$.

Finally, we prove the uniqueness of the representation (3). If (3) holds for two pairs (m, K) and (m_1, K_1), then obviously $m = \mathbb{E}(X) = m_1$, and for all t

$$\int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx)x^{-2}\mathrm{d}K(x) = \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx)x^{-2}\mathrm{d}K_1(x).$$

Differentiating this identity twice w.r.t. t (justify) we derive that for all t,

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx}\mathrm{d}K(x) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx}\mathrm{d}K_1(x).$$

The inversion theorem yields that $K = K_1$. □

Theorem 3. Let

$$\varphi(t) = \exp\left\{imt + \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx)x^{-2}\mathrm{d}K(x)\right\}, \tag{4}$$

where $m \in \mathbb{R}$ and K is a bounded distribution function. Then φ is the characteristic function of a random variable in $\mathcal{F}$.

Proof. W.l.o.g. we may assume that $m = 0$. We approximate the integral

$$I = \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx)x^{-2}\mathrm{d}K(x)$$

by

$$I_n = \sum_{k=-n^2}^{-1} I_{nk} + \sum_{k=1}^{n^2} I_{nk}$$

where for $k < 0$,

$$\begin{aligned} I_{nk} &= (\mathrm{e}^{\mathrm{i}tk/n} - 1)(k/n)^{-2}\left(K((k+1)/n) - K(k/n)\right) \\ &\quad - \mathrm{i}t(k/n)^{-1}\left(K((k+1)/n) - K(k/n)\right), \end{aligned}$$

and for $k > 0$,

$$\begin{aligned} I_{nk} &= (\mathrm{e}^{\mathrm{i}tk/n} - 1)(k/n)^{-2}\left(K(k/n) - K((k-1)/n)\right) \\ &\quad - \mathrm{i}t(k/n)^{-1}\left(K(k/n) - K((k-1)/n)\right). \end{aligned}$$

Obviously, we have just approximated a bounded, continuous integrand by simple functions and so $\lim_{n\to\infty} I_n = I$. Comparing with (1) we see that $\exp(I_{nk})$ is the characteristic function of a Poisson $\mathcal{P}(a_{nk}, b_{nk}, \lambda_{nk})$ random variable X_{nk} (what are $a_{nk}, b_{nk}, \lambda_{nk}$?). Taking X_{nk} independent and setting

$$X_n := \sum_{k=-n^2, k\neq 0}^{n^2} X_{nk},$$

we see that X_n are infinitely divisible and that $\varphi_{X_n}(t) = \exp(I_n)$. Hence, for all t, $\lim_{n\to\infty} \varphi_{X_n}(t) = \varphi(t)$. Since $\varphi(t)$ is continuous at 0 this proves that $\varphi(t)$ is the characteristic function of some random variable X, and that $X_n \to X$ in law. By Theorem 14-2 D, X is infinitely divisible. Since $\varphi(t)$ is twice continuously differentiable, X has finite variance. $\square$

It follows from Theorems 2 and 3 that each $X \in \mathcal{F}$ is uniquely characterized by a pair (m, K), where $m \in \mathbb{R}$ is the expectation of X and K is a bounded distribution function. The pairs of two random variables are equal iff the variables are identically distributed.

The proof of Theorem 3 also yields that every $X \in \mathcal{F}$ is a weak limit of a sequence of sums of independent Poisson random variables.

Theorem 4. Let X_n be a sequence in $\mathcal{F}$ such that $\sup_n \mathrm{Var}(X_n) < \infty$. Then the following statements are equivalent.
(A) $X_n \to X \in \mathcal{F}$ with the pair (m, K).
(B) If (m_n, K_n) is the pair associated to X_n, then $m_n \to m$ and $K_n \to K$ weakly.
If (A)-(B) hold, then $\mathrm{Var}(X_n) \to \mathrm{Var}(X)$ iff $K_n \to K$ completely.

Proof. The last statement is obvious since $\mathrm{Var}(X_n) = K_n(\infty)$ and $\mathrm{Var}(X) = K(\infty)$.
(B) $\Rightarrow$ (A) If $K_n \to K$ weakly, then by the third Helly theorem,

$$\lim_{n\to\infty} \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx)x^{-2}\mathrm{d}K_n(x) = \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx)x^{-2}\mathrm{d}K(x).$$

Hence, $\lim_{n\to\infty} \varphi_{X_n}(t) = \varphi_X(t)$, where $X \in \mathcal{F}$ is specified by the pair (m, K), and $X_n \to X$ weakly.
(A) $\Rightarrow$ (B) It follows from Theorem 1 that $m_n \to m$. Since $\varphi_{X_n}(t) \to \varphi_X(t)$, we have that $\log \varphi_{X_n}(t) \to \log \varphi_X(t)$, and so

$$\lim_{n\to\infty} \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx)x^{-2}\mathrm{d}K_n(x) = \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx)x^{-2}\mathrm{d}K(x).$$

Assume now that K_n does not converge to K weakly. Then, by Helly's compactness theorem ($K_n(\infty) = \mathrm{Var}(X_n) \le C$), there is a subsequence K_{n_j} such that $K_{n_j} \to \tilde{K}$ weakly, and $\tilde{K} \neq K$. By the third Helly theorem,

$$\lim_{j\to\infty} \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx)x^{-2}\mathrm{d}K_{n_j}(x) = \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx)x^{-2}\mathrm{d}\tilde{K}(x),$$

and so for all t,

$$\int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx)x^{-2}\mathrm{d}\tilde{K}(x) = \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx)x^{-2}\mathrm{d}K(x).$$

Differentiating twice w.r.t. t we obtain

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx}\mathrm{d}\tilde{K}(x) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx}\mathrm{d}K(x). \tag{5}$$

The relation (5) and the inversion theorem yield that $\tilde{K} = K$, a contradiction. Hence we must have that $K_n \to K$ weakly. $\square$