MATH 589

The Central Limit Theorems

In these notes I will sketch without proofs the solution to the general Central Limit Problem. The ideas and the arguments behaind the solution are similar to the bounded variance case. The proofs however are rather lengthy, and this year I will skip them (instead, we will cover in the course the ergodic/information theory).

The first thing one needs is a general representation theorem for infinitely divisible characteristic functions.

Theorem 1A. If X is an infinitely divisible random variable, then there exists a unique real number γ and a unique bounded distribution function G such that

$$\varphi_X(t) = \exp\left\{ \mathrm{i}\gamma t + \int_{\mathbb{R}} \left[\mathrm{e}^{\mathrm{i}tx} - 1 - \frac{\mathrm{i}tx}{1+x^2} \right] \frac{1+x^2}{x^2} \mathrm{d}G(x) \right\},$$

where the integrand is defined by continuity at $x = 0$ (and equals $-t^2/2$).

Theorem 1B. Let

$$\varphi(t) = \exp\left\{ \mathrm{i}\gamma t + \int_{\mathbb{R}} \left[\mathrm{e}^{\mathrm{i}tx} - 1 - \frac{\mathrm{i}tx}{1+x^2} \right] \frac{1+x^2}{x^2} \mathrm{d}G(x) \right\},$$

where $\gamma \in \mathbb{R}$, and G is a bounded distribution function. Then φ is the characteristic function of an infinitely divisible random variable.

The pair (γ, G) is called the Lévy-Khintchine pair of the random variable X. A normal $\mathcal{N}(m, \sigma^2)$ random variable has the Lévy-Khintchine pair $\gamma = m$ and $G(x) = 0$ for $x < 0$, $G(x) = \sigma^2$ for $x \geq 0$. A Poisson $\mathcal{P}(\lambda)$ random variable has the Lévy-Khintchine pair $\gamma = \lambda/2$ and $G(x) = 0$ for $x < 1$, $G(x) = \lambda/2$ for $x \geq 1$. Note that in this case $\gamma \neq \mathbb{E}(X) = \lambda$!

An infinitely divisible random variable has finite variance iff $\int_{\mathbb{R}} x^2 G(x) \mathrm{d}x < \infty$. In this case (γ, G) and (m, K) are related by

$$m = \gamma + \int_{\mathbb{R}} x \mathrm{d}G(x), \qquad K(x) = \int_{-\infty}^{x} (1 + y^2) \mathrm{d}G(y).$$

By approximating the integrand in with simple functions (like in the bounded variance case), one shows that any infinitely divisible random variable is a limit of a sequence of sums of independent Poisson random variables.

Recall that if X_n are infinitely divisible and $X_n \to X$, then X is also infinitely divisible.

The second thing one needs is the convergence theorem.

Theorem 2. Let X_n be a sequence of infinitely divisible random variables. Then the following statements are equivalent.
(A) $X_n \to X$ with the pair (γ, G).
(B) If (γ_n, G_n) is the pair of X_n, then $\gamma_n \to \gamma$ and $G_n \to G$ completely.

We now turn to the Central Limit Problem/Theorems. Let X_{nk}, $1 \le k \le k_n$, $k_n \to \infty$, be a double sequence of row-wise independent random variables. The sequence is called infinitesimal if for all $\epsilon > 0$,

$$\lim_{n\to\infty} \max_{1\le k\le k_n} P(|X_{nk}| > \epsilon) = 0.$$

If X_{nk} have finite variance and $\underline{\sigma} = 0$, then X_{nk} is an infinitesimal system. Let

$$S_n = \sum_{k=1}^{k_n} X_{nk}.$$

The resolution of the first part of the Central Limit Problem is given by

Theorem 3. **(A)** If X_{nk} is an infinitesimal double sequence such that for some sequence c_n of constants $S_n - c_n$ converges to a random variable X, then X is infinitely divisible.
(B) If X is an infinitely divisible random variable, then there is an infinitesimal double sequence X_{nk} such that $S_n \to X$ weakly.

The proof of part (B) is again simple while the proof of (A) requires some effort.

Theorem 3 resolves the first part of the Central Limit Problem under the very natural infinitesimality condition: The set of all possibile limit laws of the sums S_n coincides with the set of all infinitely divisible distributions.

Concerning the second part, the conditions for convergence to any specific infinitely divisible distribution can be found using Theorem 2. Even an informal description of the complete theory is rather lengthy. Instead, I will focus on several examples.

First, we will discuss convergence to a normal random variable. For $\tau > 0$ we set

$$a_{nk}(\tau) = \int_{|x|<\tau} x \mathrm{d}\mu_{X_{nk}}, \qquad \sigma_{nk}^2(\tau) = \int_{|x|<\tau} x^2 \mathrm{d}\mu_{X_{nk}} - \left(\int_{|x|<\tau} x \mathrm{d}\mu_{X_{nk}}\right)^2,$$

$$a_n(\tau) = \sum_{k=1}^{k_n} a_{nk}(\tau), \qquad \sigma_n^2(\tau) = \sum_{k=1}^{n} \sigma_{nk}^2(\tau).$$

Theorem 4. Let X_{nk} be a double sequence. The following statements are equivalent.
(A) S_n converges in law to a $\mathcal{N}(m, \sigma^2)$ random variable and the sequence X_{nk} is infinitesimal.
(B) $a_n(\tau) \to m$ and $\sigma_n^2(\tau) \to \sigma^2$ for some $\tau > 0$, and for every $\epsilon > 0$,

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} P(|X_{nk}| > \epsilon) = 0.$$

The above result has an important corollary.

Theorem 5. Let X_{nk} be a double sequence and assume that $S_n \to X$ weakly. Then the following statements are equivalent.
(A) X is a normal random variable and the sequence X_{nk} is infinitesimal.
(B) For every $\epsilon > 0$,

$$\lim_{n\to\infty} P(\max_{1\le k\le k_n} |X_{nk}| > \epsilon) = 0.$$

Concerning the convergence to the Poisson law, we have

Theorem 6. Let X_{nk} be an infinitesimal system. Then the following statements are equivalent.
(A) S_n converges in law to a Poisson $\mathcal{P}(\lambda)$ random variable.
(B) $a_n(\tau) \to 0$ and $\sigma_n^2(\tau) \to 0$ for some $\tau \in (0,1)$, and for all $\epsilon \in (0,1)$,

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{\substack{|x|>\epsilon \\ |x-1|>\epsilon}} \mathrm{d}\mu_{X_{nk}} = 0, \qquad \lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{|x-1|<\epsilon} \mathrm{d}\mu_{X_{nk}} = \lambda.$$

Finally, we briefly discuss the weak law of large numbers and the Classical degenerate convergence condition. Theorem 4 is applicable to the degenerate case $\sigma^2 = 0$ and gives the conditions that $S_n \to m$ weakly=in probability. It is an instructive exercise to show that Theorem 4 in fact implies the Classical degenerate convergence condition discussed in the previous lecture notes!

The last result I would like to mention is Theorem 14-6B in Burrill's book, which extends the Classical degenerate convergence condition to double sequences X_{nk}.

Theorem 7. Let X_{nk} be a double sequence and μ_{nk} a median of X_{nk}. Then the following statements are equivalent.
(A) There exists a sequence c_n of constants such that $S_n - c_n$ converges in probability to

zero.

(B)

$$\lim_{n\to\infty}\int_{|x|>1} \mathrm{d}\mu_{X_{nk}}(x+\mu_{nk}) = 0, \qquad \lim_{n\to\infty}\int_{|x|\le 1} x^2 \mathrm{d}\mu_{X_{nk}}(x+\mu_{nk}) = 0.$$

Of course, one may ask what c_n's are! They are given by

$$c_n = \sum_{k=1}^{k_n} \big(\mu_{nk} + \mathbb{E}((X_{nk} - \mu_{nk})K_{\{|X_{nk}-\mu_{nk}|\le 1\}})\big) + r_n,$$

where r_n is any sequence converging to zero.