MATH 589

Notes on the bounded variance Central Limit Theorem

Let X_{nk}, $n \geq 1$, $1 \leq k \leq k_n$, $k_n \to \infty$, be a double sequence of row independent random variables. We assume that the random variables have finite variances and set

$$\sigma_{nk}^2 = \mathrm{Var}(X_{nk}), \qquad \sigma_n = \left(\sum_{k=1}^{n} \sigma_{nk}^2\right)^{1/2},$$

$$\underline{\sigma} = \limsup_{n\to\infty} \max_{1\leq k\leq k_n} \sigma_{nk}^2, \qquad \overline{\sigma} = \limsup_{n\to\infty} \sigma_n.$$

We are interested in the Central Limit Problem under the conditions $\underline{\sigma} = 0$ and $\overline{\sigma} < \infty$. $\underline{\sigma} = 0$ is an infinitesimality condition (ensuring that X_{nk} are uniformly small); $\overline{\sigma} < \infty$ is the bounded variance condition. Recall that these conditions already appeared in our discussion of the Lindberg-Feller theorem.

Let

$$S_n = \sum_{k=1}^{k_n} X_{nk}. \tag{1}$$

Note that $\mathrm{Var}(S_n) = \sigma_n^2$. Recall that $\mathcal{F}$ is a family of infinitely divisible random variables with finite variance.

Theorem 1. (A) Suppose that $\underline{\sigma} = 0$, $\overline{\sigma} < \infty$, and that $S_n \to X$ weakly. Then $X \in \mathcal{F}$.
(B) Assume that $X \in \mathcal{F}$. Then there is a double sequence X_{nk} satisfying $\underline{\sigma} = 0$ and $\overline{\sigma} < \infty$ such that $S_n \to X$ weakly.

Theorem 1 resolves the first part of the Central Limit Problem under the conditions $\underline{\sigma} = 0, \overline{\sigma} < \infty$: The set of all possible limit laws of the sums S_n coincides with the set of all infinitely divisible distributions with finite variance.

The proof of Theorem 1 (B) is elementary. For each n we can write

$$X = X_{n1} + \cdots + X_{nn},$$

where X_{nk} are i.i.d. Since $\sigma_{nk}^2 = \mathrm{Var}(X)/n$, $\underline{\sigma} = 0$ and $\overline{\sigma} = \sqrt{\mathrm{Var}(X)}$.

Before we prove Theorem 1 (A) we collect some preliminary results.

Let

$$m_{nk} := \mathbb{E}(X_{nk}), \qquad m := \mathbb{E}(S_n) = \sum_{k=1}^{k_n} m_{nk}.$$

We denote by φ_n the characteristic function of $S_n - m_n$ and by φ_{nk} the characteristic function of $X_{nk} - m_{nk}$. Similarly, μ_n stands for the distribution of $S_n - m_n$ and μ_{nk} for the distribution of $X_{nk} - m_{nk}$.

Obviously,

$$\sigma_{nk}^2 = \int_{\mathbb{R}} x^2 \mathrm{d}\mu_{nk}, \qquad \int_{\mathbb{R}} x \mathrm{d}\mu_{nk} = 0, \qquad \mu_{nk}(\,\cdot\,) = \mu_{X_{nk}}(\,\cdot + m_{nk}),$$

$$\varphi_n(t) = \prod_{k=1}^{n} \varphi_{nk}(t), \qquad \varphi_{S_n}(t) = \mathrm{e}^{\mathrm{i}tm_n}\varphi_n(t).$$

Proposition 1. Assume that $\underline{\sigma} = 0$ and let t be given. Then

$$\lim_{n\to\infty} \max_{1\le k\le k_n} |\varphi_{nk}(t) - 1| = 0. \tag{2}$$

In particular, $\mathrm{Ln}\varphi_{nk}(t)$ is defined for n large enough. If in addition $\overline{\sigma} < \infty$, then

$$\lim_{n\to\infty} \left(\sum_{k=1}^{n} \mathrm{Ln}\varphi_{nk}(t) - \sum_{k=1}^{n} (\varphi_{nk}(t) - 1) \right) = 0. \tag{3}$$

Remark. We have already proven this result during the discussion of the Lindberg-Feller theorem!

Proof.

$$\begin{aligned} |\varphi_{nk}(t) - 1| &= \left| \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}tx) \mathrm{d}\mu_{nk} \right| \\ &\le \frac{t^2}{2} \int_{\mathbb{R}} x^2 \mathrm{d}\mu_{nk} = (t^2\sigma_{nk}^2)/2. \end{aligned}$$

Hence,

$$\lim_{n\to\infty} \max_{1\le k\le k_n} |\varphi_{nk}(t) - 1| \le (t^2\underline{\sigma})/2 = 0.$$

Starting with n large enough, $\max_{1\le k\le k_n} |\varphi_{nk}(t) - 1| < 1/2$. For such n,

$$\sum_{k=1}^{n} \mathrm{Ln}\varphi_{nk}(t) = \sum_{k=1}^{n} (\varphi_{nk}(t) - 1) + \sum_{k=1}^{n} R_{nk}(t), \tag{4}$$

where

$$R_{nk}(t) = \sum_{j=2}^{\infty} \frac{(-1)^{j-1}}{j} (\varphi_{nk}(t) - 1)^j. \tag{5}$$

We estimate

$$
\begin{aligned}
|R_{nk}(t)| &\le \frac{1}{2}\sum_{j=2}^{\infty} |\varphi_{nk}(t)-1|^j \\
&= \frac{1}{2}\frac{|\varphi_{nk}(t)-1|^2}{1-|\varphi_{nk}(t)-1|} \\
&\le |\varphi_{nk}(t)-1|^2 \\
&= |\varphi_{nk}(t)-1| \left| \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}tx}-1)\mathrm{d}\mu_{nk} \right|,
\end{aligned}
$$

and so

$$
|R_{nk}(t)| \le \frac{t^2}{2}|\varphi_{nk}(t)-1| \int_{\mathbb{R}} x^2 \mathrm{d}\mu_{nk}. \tag{6}
$$

Let

$$
R_n(t) := \sum_{k=1}^{k_n} R_{nk}(t).
$$

The estimate (6) yields

$$
|R_n(t)| \le \sum_{k=1}^{k_n} |R_{nk}(t)| \le (\overline{\sigma} t)^2 \max_{1\le k\le k_n} |\varphi_{nk}(t)-1|/2.
$$

The relation (2) yields that

$$
\lim_{n\to\infty} R_n(t) = 0, \tag{7}
$$

and the statement follows. □

Note that

$$
\begin{aligned}
\sum_{k=1}^{k_n}(\varphi_{nk}(t)-1) &= \sum_{k=1}^{k_n}\int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}tx}-1-\mathrm{i}tx)\mathrm{d}\mu_{nk} \\
&= \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}tx}-1-\mathrm{i}tx)x^{-2}\mathrm{d}K_n(x),
\end{aligned}
$$

where

$$
K_n(x) = \sum_{k=1}^{k_n}\int_{-\infty}^{x} y^2 \mathrm{d}\mu_{nk}(y) = \sum_{k=1}^{k_n}\int_{-\infty}^{x} y^2 \mathrm{d}\mu_{X_{nk}}(y+m_{nk}). \tag{8}
$$

Note also that $K_n(\infty) = \sigma_n^2$.

Let $Y_n \in \mathcal{F}$ be a random variable associated to the pair (m_n, K_n). Obviously

$$\varphi_{Y_n}(t) = \exp\left\{ \mathrm{i} m_n t + \sum_{k=1}^{k_n} (\varphi_{nk}(t) - 1) \right\},$$

and

$$\mathbb{E}(Y_n) = m_n, \qquad \mathrm{Var}(Y_n) = \sigma_n^2.$$

Proposition 2. Suppose that $\underline{\sigma} = 0$ and $\overline{\sigma} < \infty$. Then $S_n \to X$ weakly iff $Y_n \to X$ weakly.

Proof. $S_n \to X$ weakly iff for all t,

$$\lim_{n\to\infty} \varphi_{S_n}(t) = \varphi_X(t).$$

Hence, $S_n \to X$ weakly iff for all t,

$$\lim_{n\to\infty} \mathrm{e}^{\mathrm{i} m_n t} \varphi_n(t) = \varphi_X(t).$$

Hence, $S_n \to X$ weakly iff for all t,

$$\lim_{n\to\infty} \exp\left\{ \mathrm{i} m_n t + \sum_{k=1}^{k_n} \mathrm{Ln} \varphi_{nk}(t) \right\} = \varphi_X(t).$$

It follows from Proposition 2 that $S_n \to X$ weakly iff for all t,

$$\lim_{n\to\infty} \exp\left\{ \mathrm{i} m_n t + \sum_{k=1}^{k_n} (\varphi_{nk}(t) - 1) \right\} = \varphi_X(t),$$

and so $S_n \to X$ weakly iff for all t

$$\lim_{n\to\infty} \varphi_{Y_n}(t) = \varphi_X(t).$$

This yields the statement. □

Proof of Theorem 1 (A). Proposition 2 yields that $Y_n \to X$. Obviously, $\sup_n \mathrm{Var}(Y_n) < \infty$ and Theorem 1 from the previous lecture notes yields that $X \in \mathcal{F}$. □

Concerning the resolution of the second part of the Central Limit Problem, the above argument provides a key ingredient. Combining Proposition 2 with Theorem 4 in the previous lecture notes we derive:

Theorem 2. Suppose that $\underline{\sigma} = 0$ and $\overline{\sigma} < \infty$. Then $S_n \to X$ with the pair (m, K) if and only if $m_n \to m$ and $K_n \to K$ weakly. Moreover, $\sigma_n^2 \to \mathrm{Var}(X)$ iff $K_n \to K$ completely.

Theorem 2 allow us to identify conditions for the convergence to a particular law in the family $\mathcal{F}$. We will only discuss the convergence to normal and Poisson random variables.

In the normal case we expect to recover the Lindberg-Feller theorem. In fact, the next theorem is obviously a natural generalization of the Lindberg-Feller theorem.

Theorem 3. Let X_{nk} be a double sequence of random variables such that $\sigma_n^2 \to \sigma^2$. Then the following statements are equivalent.
(A) S_n converges weakly to a $\mathcal{N}(m, \sigma^2)$ random variable and $\underline{\sigma} = 0$.
(B) $\lim_{n\to\infty} m_n = m$, and for every $\epsilon > 0$,

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{|x-m_{nk}|>\epsilon} (x - m_{nk})^2 \mathrm{d}\mu_{X_{nk}}(x) = 0. \tag{9}$$

Proof. The pair associated to a $\mathcal{N}(m, \sigma^2)$ random variable is obviously (m, K), where $K(x) = 0$ if $x < 0$ and $K(x) = \sigma^2$ if $x \geq 0$. Let K_n be defined by (8). Since $K_n(\infty) = \sigma_n^2$,

$$\sigma_n^2 \to \sigma^2 \;\Leftrightarrow\; K_n(\infty) \to \sigma^2. \tag{10}$$

(A) $\Rightarrow$ **(B)** By Theorem 2, if S_n converges in law to a $\mathcal{N}(m, \sigma^2)$ random variable, then $m_n \to m$ and

$$\lim_{n\to\infty} K_n(x) = K(x) \qquad \text{for } x \neq 0. \tag{11}$$

The relations (10) and (11) imply that for $\epsilon > 0$,

$$\lim_{n\to\infty} K_n(-\epsilon) = 0, \qquad \lim_{n\to\infty} (K_n(\infty) - K_n(\epsilon)) = 0, \tag{12}$$

and so

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{|x|>\epsilon} x^2 \mathrm{d}\mu_{X_{nk}}(x + m_{nk}) = 0. \tag{13}$$

Obviously, (13)=(9).
(B) $\Rightarrow$ **(A)** As in the proof of the Lindberg-Feller theorem, (9) implies that $\underline{\sigma} = 0$. Moreover, since (13)=(9) implies (12) which is equivalent to (11), the result follows from

Theorem 2. □

Theorem 4. Let X_{nk} be a double sequence of random variables satisfying $\underline{\sigma} = 0$ and $\sigma_n^2 \to \lambda$. Then S_n converges weakly to a Poisson $\mathcal{P}(\lambda)$ random variable if and only if $m_n \to \lambda$ and for every $\epsilon > 0$,

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{|x-m_{nk}-1|>\epsilon} (x - m_{nk})^2 \mathrm{d}\mu_{X_{nk}}(x) = 0. \tag{14}$$

Proof. The pair associated to a $\mathcal{P}(\lambda)$ random variable is (λ, K), where $K(x) = 0$ if $x < 1$ and $K(x) = \lambda$ if $x \geq 1$ (show that!). Let $K_n(x)$ be defined by (8). Then $K_n(\infty) = \sigma_n^2 \to \lambda$. By Theorem 2, S_n converges weakly to a $\mathcal{P}(\lambda)$ random variable if and only if $m_n \to \lambda$ and

$$\lim_{n\to\infty} K_n(x) = K(x) \qquad \text{for} \qquad x \neq 1, \tag{15}$$

and to prove the statement it suffices to show that (14) $\Leftrightarrow$ (15). Note that relation (15) holds if and only if for every $\epsilon > 0$,

$$\lim_{n\to\infty} K_n(1-\epsilon) = 0, \qquad \lim_{n\to\infty} (K_n(\infty) - K_n(1+\epsilon)) = 0,$$

that is, if and only if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \sum_{n=1}^{k_n} \int_{|x-1|>\epsilon} x^2 \mathrm{d}\mu_{X_{nk}}(x + m_{nk}) = 0. \tag{16}$$

Obviously, (16)=(14). □