MATH 743
SPECTRAL THEORY OF ANDERSON TYPE HAMILTONIANS
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LECTURE 8

RANK TWO Anderson Type Hamiltonians

O.1 References

[Aiz] Aizenman M.: Localization at weak disorder: Some elementary bounds, Rev. Math. Phys. 6, 1163-1182, 1994.

[AM] Aizenman M., Molchanov S.: Localization at large disorder and at exterme energies: An elementary derivation, Commun. Math. Phys. 157, 245-278, 1993.

[Gr] Graf G.M.: Anderson localization and the space-time characteristic of continuum states, J. Stat. Phys. 75, 1994.

[Si] Simon B.: Spectral analysis of rank one perturbations and applications, CRM Lecture Notes Vol. 8, pp. 109-149, AMS, Providence, RI, 1995.

[DJLS] del Rio R, Jitomirskaya S., Last Y., Simon B.: Operators with singular continuous spectrum, IV. Hausdorff dimensions, rank one perturbations, and localization, J. d'Analyse Math. 69 (1996), 153-200, available at
http://www.math.caltech.edu/people/biblio.html

[JL1] Jaksic V., Last Y.: Spectral structure of Anderson type Hamiltonians, Invent. Math. 141, (2000) 561-577, available on the course web site.

[JL2] Jaksic V., Last Y.: Simplicity of singular spectrum in Anderson type Hamiltonians, preprint.

[Mol] Molchanov S.: Lectures on Random Media, in *Lectures on Probability*, ed. P. Bernard, Lecture Notes in Mathematics, 1581, Springer-Verlag 1994.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space, H_0 a selfadjoint operator on $\mathcal{H}$ and $\psi, \varphi \in \mathcal{H}$ two unit vectors. The family of operators

$$H_{\lambda,\eta} = H_0 + \lambda(\psi|\cdot)\psi + \eta(\varphi|\cdot)\varphi, \qquad \lambda, \eta \in \mathbb{R},$$

we will call rank two Anderson type Hamiltonians. This class of operators is slightly more general than the Anderson type Hamiltonians of Lecture 6 in the case $\Gamma = \{1, 2\}$ since we do not require that H_0 is bounded or that $\psi \perp \varphi$.

The cyclic subspace spanned by $H_{\lambda,\eta}$ and $\{\psi, \varphi\}$ is independent of λ and η and equal to the cyclic subspace spanned by H_0 and $\{\psi, \varphi\}$. In this lecture, w.l.o.g. we may assume that $\{\psi, \varphi\}$ is a cyclic set of vectors for H_0.

We denote by $\mathcal{H}_{\lambda,\eta,\psi}$ and $\mathcal{H}_{\lambda,\eta,\varphi}$ the cyclic subspaces spanned by $H_{\lambda,\eta}$ and ψ, φ respectively. In the first part of the lecture, we will study the relation between $H_{\lambda,\eta} \restriction \mathcal{H}_{\lambda,\eta,\psi}$ and $H_{\lambda,\eta} \restriction \mathcal{H}_{\lambda,\eta,\varphi}$. In the second part of the lecture we will discuss the rank two Wegner type estimate.

2. Jaksic-Last theorems III

Proposition 2.1. The following statements are equivalent:
(1) For all $\lambda, \eta \in \mathbb{R}$ the subspaces $\mathcal{H}_{\lambda,\eta,\psi}$ and $\mathcal{H}_{\lambda,\eta,\varphi}$ are not orthogonal.
(2) For all $\lambda, \eta \in \mathbb{R}$ there exists $z \in \mathbb{C} \setminus \mathbb{R}$ such that $(\psi|(H_{\lambda,\eta} - z)^{-1}\varphi) \neq 0$.
(3) The subspaces $\mathcal{H}_{0,0,\psi}$ and $\mathcal{H}_{0,0,\varphi}$ are not orthogonal.
(4) There exists $z \in \mathbb{C} \setminus \mathbb{R}$ such that $(\psi|(H_0 - z)^{-1}\varphi) \neq 0$.
Moreover, in (2) and (4) we can replace $z \in \mathbb{C} \setminus \mathbb{R}$ with $z \in \mathbb{C}_+$.

The proof of Proposition 2.1 is simple and is left for homework.

The minimal assumption under which one may say something about the relation between the operators $H_{\lambda,\eta} \restriction \mathcal{H}_{\lambda,\eta,\psi}$ and $H_{\lambda,\eta} \restriction \mathcal{H}_{\lambda,\eta,\varphi}$ is:

Assumption 2.2. For all $\lambda, \eta \in \mathbb{R}$ the subspaces $\mathcal{H}_{\lambda,\eta,\psi}$ and $\mathcal{H}_{\lambda,\eta,\varphi}$ are not orthogonal.

The following theorem was proven in [JL1]:

Theorem 2.3. Suppose that Assumption 2.2 holds. Then the operators $H_{\lambda,\eta} \restriction \mathcal{H}_{\lambda,\eta,\psi}$ and $H_{\lambda,\eta} \restriction \mathcal{H}_{\lambda,\eta,\varphi}$ are unitarily equivalent for Lebesgue a.e. $(\lambda, \eta) \in \mathbb{R}^2$.

Proof. We denote by $\mu_\psi^{\lambda,\eta}$ and $\mu_\varphi^{\lambda,\eta}$ the spectral measures for $H_{\lambda,\eta}$ and ψ, φ respectively. Theorem 2.3 holds iff $\mu_\psi^{\lambda,\eta}$ and $\mu_\varphi^{\lambda,\eta}$ are equivalent measures for Lebesgue a.e. $(\lambda, \eta) \in \mathbb{R}^2$ (see Section VII.2 of Reed-Simon I and the homework). Let

$$\mu_\psi^{\lambda,\eta} = f_{\psi,\varphi}^{\lambda,\eta}\mu_\varphi^{\lambda,\eta} + \nu_{\psi,\varphi}^{\lambda,\eta}$$

be the Radon-Nikodym decomposition and

$$S = \{(\lambda, \eta) \in \mathbb{R}^2 \, : \, \nu_{\psi,\varphi}^{\lambda,\eta} \neq 0\}.$$

Clearly, $(\lambda, \eta) \in S$ iff $\mu_\psi^{\lambda,\eta}$ is not absolutely continuous w.r.t. $\mu_\varphi^{\lambda,\eta}$. Let $S_\eta = \{\lambda \in \mathbb{R} : (\lambda, \eta) \in S\}$. Then by Corollary 12.2 in Lecture 7, for all $\eta \in \mathbb{R}$ the set S_η has zero Lebesgue measure (to derive this apply Corollary 12.2 to the operator $H_0 + \eta(\varphi|\cdot)\varphi$ instead of H_0). By the Fubini theorem, the set S has zero Lebesgue measure. □

We denote by $\mathcal{H}_{\lambda,\eta,\psi,\text{sing}}$ and $\mathcal{H}_{\lambda,\eta,\varphi,\text{sing}}$ the subspaces associated respectively to the singular spectrum of $H_{\lambda,\eta} \restriction \mathcal{H}_{\lambda,\eta,\psi}$ and $H_{\lambda,\eta} \restriction \mathcal{H}_{\lambda,\eta,\varphi}$. By the previous theorem, the operators $H_{\lambda,\eta} \restriction \mathcal{H}_{\lambda,\eta,\psi,\text{sing}}$ and $H_{\lambda,\eta} \restriction \mathcal{H}_{\lambda,\eta,\varphi,\text{sing}}$ are unitarily equivalent for Lebesgue a.e. $(\lambda, \eta) \in \mathbb{R}^2$. In fact, more is true: these two operators *coincide* for Lebesgue a.e. $(\lambda, \eta) \in \mathbb{R}^2$. The next theorem is proven in [JL2]:

Theorem 2.4. Suppose that Assumption 2.2 holds. Then

$$\mathcal{H}_{\lambda,\eta,\psi,\text{sing}} = \mathcal{H}_{\lambda,\eta,\varphi,\text{sing}}$$

for Lebesgue a.e. $(\lambda, \eta) \in \mathbb{R}^2$.

Proof. By Theorem 2.3, for Lebesgue a.e. $(\lambda, \eta) \in \mathbb{R}^2$, the measures $\mu_\psi^{\lambda,\eta}$ and $\mu_\varphi^{\lambda,\eta}$ are equivalent. It follows from Fubini's theorem and Theorem 12.3 in Lecture 7 that for Lebesgue a.e. $\eta \in \mathbb{R}$,

$$\mathbf{1}_{\text{sing}}(H_{\lambda,\eta})\varphi \in \mathcal{H}_{\lambda,\eta,\psi,\text{sing}}$$

for Lebesgue a.e. $\lambda \in \mathbb{R}$. Hence, by Fubini's theorem again, for Lebesgue a.e. $(\lambda, \eta) \in \mathbb{R}^2$,

$$\mathcal{H}_{\lambda,\eta,\varphi,\text{sing}} \subset \mathcal{H}_{\lambda,\eta,\psi,\text{sing}}.$$

By reversing the roles of φ and ψ we derive the statement. $\square$

Let $\mathcal{H}_{\lambda,\eta,\text{sing}}$ be the spectral subspace associated to the singular spectrum of $H_{\lambda,\eta}$. Theorem 2.4. yields that

$$\mathcal{H}_{\lambda,\eta,\text{sing}} = \mathcal{H}_{\lambda,\eta,\psi,\text{sing}} = \mathcal{H}_{\lambda,\eta,\varphi,\text{sing}}$$

for Lebesgue a.e. $(\lambda, \eta) \in \mathbb{R}^2$. In particular, ψ is a cyclic vector for $\mathcal{H}_{\lambda,\eta,\text{sing}}$ for Lebesgue a.e. $(\lambda, \eta) \in \mathbb{R}^2$ (and of course the same holds for φ).

In the case of ac spectral subspaces we have the following result. Let $\mathcal{H}_{0,0,\text{ac}}$ be the spectral subspace associated to the ac spectrum of H_0.

Theorem 2.5. Suppose that Assumption 2.2 holds. Then the following statements are equivalent:
(1) $H_0 \restriction \mathcal{H}_{0,0,\text{ac}}$ has a cyclic vector.
(2) $\mathcal{H}_{\lambda,\eta,\psi,\text{ac}} = \mathcal{H}_{\lambda,\eta,\varphi,\text{ac}}$ for Lebesgue a.e. $(\lambda, \eta) \in \mathbb{R}^2$.

The proof of Theorem 2.5 is left for homework.

3. AMG Wegner type estimate

Let $\mathrm{d}P(\lambda, \eta) = \mathrm{d}P_1(\lambda) \otimes \mathrm{d}P_2(\eta)$ be a Borel probability measure on $\mathbb{R}^2$. In this subsection we will prove:

Theorem 3.1. Assume that P_1 and P_2 are τ-regular with constants $M_{1\tau}$ and $M_{2\tau}$. Then for $0 \leq s < \tau$,

$$\sup_{z\in\mathbb{C}\setminus\mathbb{R}} \mathbb{E}\left|(\psi|(H_{\lambda,\eta}-z)^{-1}\varphi)\right|^s \leq \frac{4\tau}{\tau-s}\max(M_{1\tau}^{s/\tau}, M_{2\tau}^{s/\tau}). \tag{1}$$

Theorem 3.1 was first proven by Aizenman and Molchanov [AM], and played an important role in first versions of the Aizenman-Molchanov theory. A transparent proof of Theorem 3.1 was given by Graf [Gr], and we will call (1) the Aizenman-Molchanov-Graf (AMG) Wegner type estimate.

It is important to add that Theorem 3.1 is actually not needed for the Aizenman-Molchanov theory (although, on the first sight, it played a key role in the arguments of [AM] and [Gr]). This was quickly realized (after the publication of [AM] and [Gr]) in [Aiz], [Mol] and [Si]. Hence, we will not make use of this theorem in Lecture 10 where we will discuss the Aizenman-Molchanov theory. Nevertheless, Theorem 3.1 is an important structural result which is often technically useful in study of Anderson type Hamiltonians.

The proof below follows closely [Gr].

Proof of Theorem 3.1. W.l.o.g. we may suppose that Assumption 2.2 holds (otherwise, Theorem 3.1 trivially holds).

Set

$$G_{\lambda,\eta}(\psi,\psi,z) := (\psi|(H_{\lambda,\eta}-z)^{-1}\psi), \qquad G_{\lambda,\eta}(\psi,\varphi,z) := (\psi|(H_{\lambda,\eta}-z)^{-1}\varphi),$$

$b_{11} := (\psi|(H_0-z)^{-1}\psi)$, $b_{12} := (\psi|(H_0-z)^{-1}\varphi)$, $b_{21} = (\varphi|(H_0-z)^{-1}\psi)$, $b_{22} = (\varphi|(H_0-z)^{-1}\varphi)$. W.l.o.g. we may assume that $b_{12} \neq 0$ and $b_{21} \neq 0$ (the set of z's such that $b_{12} = 0$ or $b_{21} = 0$ is at most countable). Note that

$$\begin{aligned}(H_{\lambda,\eta}-z)^{-1} - (H_0-z)^{-1} &= (H_{\lambda,\eta}-z)^{-1}(H_0 - H_{\lambda,\eta})(H_0-z)^{-1} \\ &= -(H_{\lambda,\eta}-z)^{-1}\left[\lambda(\psi|\cdot)\psi + \eta(\varphi|\cdot)\varphi\right](H_0-z)^{-1}.\end{aligned}$$

Substituting this identities in $(\psi|\cdot\varphi)$ and $(\psi|\cdot\psi)$ we obtain a system:

$$\begin{aligned}G_{\lambda,\eta}(\psi,\varphi,z)(1+\eta b_{22}) + G_{\lambda,\eta}(\psi,\psi,z)\lambda b_{12} &= b_{12} \\ G_{\lambda,\eta}(\psi,\varphi,z)\eta b_{21} + G_{\lambda,\eta}(\psi,\psi,z)(1+\lambda b_{11}) &= b_{11}.\end{aligned}$$

Solving this system we obtain the estimate

$$|G_{\lambda,\eta}(\psi,\varphi,z)| \leq \frac{|b_{12}|}{|(1+\lambda b_{11})(1+\eta b_{22}) - \lambda\eta b_{12}b_{21}|} \tag{2}$$

(We have equality in (2) whenever the denominator is different from zero). Let $B = [b_{ij}]$. We claim that the matrix B is invertible. Indeed, if P is the projection on the two dimensional subspace spanned by $\{\psi, \varphi\}$, then $B = P(H_0 - z)^{-1}P$ and

$$\mathrm{Im}B = \frac{1}{2\mathrm{i}}(B - B^*) = \mathrm{Im}z P(H_0 - z)^{-1}(H_0 - \overline{z})^{-1}P,$$

and so $\mathrm{Im}B/\mathrm{Im}z > 0$ for all $z \in \mathbb{C}\setminus\mathbb{R}$. Hence, B is invertible. Let $A = B^{-1} = [a_{ij}]$. Since

$$\mathrm{Im}A = \frac{1}{2\mathrm{i}}(A - A^*) = -A(\mathrm{Im}B)A^*,$$

for $z \in \mathbb{C}\setminus\mathbb{R}$, $\mathrm{Im}A/\mathrm{Im}z < 0$. It follows that $\det \mathrm{Im}A > 0$ for $z \in \mathbb{C}\setminus\mathbb{R}$. Note that

$$\det \mathrm{Im}A = \mathrm{Im}a_{11}\mathrm{Im}a_{22} + \frac{1}{2}\mathrm{Re}(a_{12}a_{21}) - \frac{1}{4}(|a_{12}|^2 + |a_{21}|^2). \tag{3}$$

By elementary linear algebra, the r.h.s. of (2) can be rewritten so that

$$|G_{\lambda,\eta}(\psi, \varphi, z)| \leq \frac{|a_{21}|}{|(\lambda + a_{11})(\eta + a_{22}) - a_{12}a_{21}|} \tag{4}$$

This formula is the starting point of our estimates. Let

$$\tilde{\lambda} = \lambda + \mathrm{Re}a_{11}, \qquad \tilde{\eta} = \eta + \mathrm{Re}a_{22}. \tag{5}$$

By retaining only the real/imaginary part of the denominator in (4), we get

$$\begin{aligned} |G_{\lambda,\eta}(\psi, \varphi, z)| &\leq \frac{|a_{21}|}{|\tilde{\lambda}\tilde{\eta} - \mathrm{Im}a_{11}\mathrm{Im}a_{22} - \mathrm{Re}(a_{12}a_{21})|} \\ |G_{\lambda,\eta}(\psi, \varphi, z)| &\leq \frac{|a_{21}|}{|\tilde{\lambda}\mathrm{Im}a_{22} + \tilde{\eta}\mathrm{Im}a_{11} - \mathrm{Im}(a_{12}a_{21})|} \end{aligned} \tag{6}$$

We shall treat first the case

$$\max(|\mathrm{Im}a_{11}|, |\mathrm{Im}a_{22}|) \leq |a_{21}|/2.$$

The formula (3) yields

$$\begin{aligned} c^2 := \mathrm{Im}a_{11}\mathrm{Im}a_{22} + \mathrm{Re}(a_{12}a_{21}) &> \frac{1}{2}(|a_{12}|^2 + |a_{21}|^2) - \mathrm{Im}a_{11}\mathrm{Im}a_{22} \\ &> |a_{21}|^2/4. \end{aligned}$$

It follows that

$$|G_{\lambda,\eta}(\psi, \varphi, z)| \leq \frac{2c}{|\tilde{\lambda}\tilde{\eta} - c^2|} = \frac{2c^{-1}}{|c^{-2}\tilde{\lambda}\tilde{\eta} - 1|}. \tag{7}$$

Let $\ell : \mathbb{R} \to \mathbb{R}$ be defined by $\ell(0) = 0$, $x\ell(x) = \min(1, x^2)$. Then for any $a, b \in \mathbb{R}$,

$$\min(|a - \ell(b)|, |b - \ell(a)|) \leq |ab - 1|. \tag{8}$$

To prove this relation, note that if $b^2 \geq 1$,

$$|a - \ell(b)| = |a - b^{-1}| \leq |b||a - b^{-1}| = |ab - 1|.$$

Similar estimate holds if $a^2 \geq 1$. On the other hand, if $a^2 < 1$ and $b^2 < 1$, then

$$(a - \ell(b))^2 = (b - \ell(a))^2 = (b - a)^2 = (ab - 1)^2 - (1 - a^2)(1 - b^2) < (ab - 1)^2.$$

Using (8) to estimate (7) (with $a = c^{-1}\tilde{\lambda}$, $b = c^{-1}\tilde{\eta}$), we derive

$$\begin{aligned} |G_{\lambda,\eta}(\psi, \varphi, z)| &\leq 2 \max\left(|\tilde{\lambda} - c\ell(c^{-1}\tilde{\eta})|^{-1}, |\tilde{\eta} - c\ell(c^{-1}\tilde{\lambda})|^{-1}\right) \\ &\leq 2(|\tilde{\lambda} - c\ell(c^{-1}\tilde{\eta})|^{-1} + |\tilde{\eta} - c\ell(c^{-1}\tilde{\lambda})|^{-1}). \end{aligned} \tag{9}$$

Since $|z_1 + z_2|^s \leq |z_1|^s + |z_2|^s$ for any two complex numbers z_1, z_2 and $0 \leq s < 1$, we derive from (9) the estimate

$$\mathbb{E}\,|G_{\lambda,\eta}(\varphi, \psi, z)|^s \leq 2 \sup_{\alpha \in \mathbb{C}} \int_{\mathbb{R}} \frac{\mathrm{d}P_1(\lambda)}{|\lambda - \alpha|^s} + 2 \sup_{\alpha \in \mathbb{C}} \int_{\mathbb{R}} \frac{\mathrm{d}P_2(\eta)}{|\eta - \alpha|^s}.$$

Estimating the r.h.s. as in Subsection 3 of Lecture 7, we derive

$$\mathbb{E}\,|G_{\lambda,\eta}(\varphi, \psi, z)|^s \leq \frac{4\tau}{\tau - s} \max(M_{1\tau}^{s/\tau}, M_{2\tau}^{s/\tau}).$$

We now handle the case

$$\max(|\mathrm{Im} a_{11}|, |\mathrm{Im} a_{22}|) \geq |a_{21}|/2.$$

We assume $|\mathrm{Im} a_{22}| \geq |a_{21}|/2$, the other subcase being similar. Then, using the second estimate in (6) we derive

$$|G_{\lambda,\eta}(\varphi, \psi, z)| \leq \frac{2}{|\tilde{\lambda} + (\tilde{\eta}\mathrm{Im} a_{11} - \mathrm{Im}(a_{12}a_{21})/\mathrm{Im} a_{22})|}$$

and so

$$\mathbb{E}\,|G_{\lambda,\eta}(\varphi, \psi, z)|^s \leq 2^s \sup_{\alpha \in \mathbb{C}} \int_{\mathbb{R}} \frac{\mathrm{d}P_1(\lambda)}{|\lambda - \alpha|^s} \leq \frac{2^s \tau}{\tau - s} M_{1\tau}^{s/\tau}.$$

Combining the cases we derive the statement. □

4. Homework.

[1] Prove Proposition 2.1.

[2] Let A be a selfadjoint operator on a Hilbert space $\mathcal{H}$, $\psi, \varphi \in \mathcal{H}$, μ_ψ, μ_φ the corresponding spectral measures, and $\mathcal{H}_\psi, \mathcal{H}_\varphi$ the corresponding spectral subspaces. Prove that the following statements are equivalent:
(1) The operators $A \restriction \mathcal{H}_\psi$ and $A \restriction \mathcal{H}_\varphi$ are unitarily equivalent.
(2) μ_ψ and μ_φ are equivalent measures.

[3] Verify carefully the applications of Fubini's theorem in Theorem 2.4.

[4] Prove Theorem 2.5.

[5] In Theorem 3.1 we have assumed that P is a product measure. Where in the proof we have used this assumption? State and prove the analog of Theorem 3.1 for a general P. Hint: The conditional Fubini theorem.