MATH 743
SPECTRAL THEORY OF ANDERSON TYPE HAMILTONIANS
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LECTURE 7

RANK ONE Anderson Type Hamiltonians

O.1 References

[Aiz] Aizenman M.: Localization at weak disorder: Some elementary bounds, Rev. Math. Phys. 6, 1163-1182, 1994.

[AG] Aizenman M., Graf G.M.: Localization bounds for an electron gas, available on mp-arc, 98-369.

[Si1] Simon B.: Spectral analysis of rank one perturbations and applications, CRM Lecture Notes Vol. 8, pp. 109-149, AMS, Providence, RI, 1995.

[Si2] Simon B.: Spectral averaging and the Krein spectral shift, Proc. Amer. Math. Soc. 126 (1998), 1409-1413, available at
http://www.math.caltech.edu/people/biblio.html

[Si3] Simon B.: Cyclic vectors in the Anderson model, Rev. Math. Phys. 6 (1994), 1183-1185

[RNS] del Rio R., Makarov N., Simon B.: Operators with singular continuous spectrum: II. Rank one operators, Commun. Math. Phys. 165 (1994), 59-67, available at
http://www.math.caltech.edu/people/biblio.html

[Gor] Gordon A.: Pure point spectrum under 1-parameter perturbations and instability of Anderson localization, Commun. Math. Phys. **164**, 489-505, 1994.

[DJLS] del Rio R, Jitomirskaya S., Last Y., Simon B.: Operators with singular continuous spectrum, IV. Hausdorff dimensions, rank one perturbations, and localization, J. d'Analyse Math. 69 (1996), 153-200, available at
http://www.math.caltech.edu/people/biblio.html

[DS] del Rio R., Simon B.: Point spectrum and mixed spectral types for rank one perturbations, Proc. Amer. Math. Soc. 125(1997), 3593-3599, available at
http://www.math.caltech.edu/people/biblio.html

[Po1] Poltoratskii A.G.: The boundary behavior of pseudocontinuable functions, St. Petersburgh Math. J. 5 (1994), 389-406, available at
http://www.math.tamu.edu/~alexeip/publications.htm

[Po2] Poltoratskii A.G.: On the Distribution of the Boundary Values of Cauchy Integrals Proc. Amer. Math. Soc., (1996), no. 8, pp. 2455–2463, available at
http://www.math.tamu.edu/~alexeip/publications.htm

[DFP] del Rio R., and Fuentes S., Poltoratskii A.G.: Coexistence of absolutely continuous and singular spectra, available at
http://www.math.tamu.edu/~alexeip/publications.htm

[JL1] Jaksic V., Last Y.: A new proof of Poltoratskii's theorem, to appear in J. Func. Anal., available on the course web site.

[JL2] Jaksic V., Last Y.: Spectral structure of Anderson type Hamiltonians, Invent. Math. 141, (2000) 561-577, available on the course web site.

[JL3] Jaksic V., Last Y.: Simplicity of singular spectrum in Anderson type Hamiltonians, preprint.

[SW] Simon B., Wolff T.: Singular continuous spectrum under rank one perturbations and localization for random Hamiltonians, Communications in Pure and Applied Mathematics 49, (1986) 75-90.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space, H_0 a selfadjoint operator on $\mathcal{H}$ and $\psi \in \mathcal{H}$ a given unit vector. The family of operators

$$H_\lambda = H_0 + \lambda(\psi|\cdot)\psi, \qquad \lambda \in \mathbb{R},$$

we will call rank one Anderson type Hamiltonians. Except for a choice of (a more convenient) notation, this class of operators coincides with the Anderson type Hamiltonians of Lecture 6 in the special case $\Gamma = \{1\}$. Note however that now we do not require that H_0 is bounded or that $\psi \in \mathrm{Dom} H_0$.

The cyclic subspace spanned by H_λ and ψ does not depend on λ and is equal to the cyclic subspace spanned by H_0 and ψ which we denote $\mathcal{H}_\psi$ (this fact is an immediate consequence of the formulas (2) below). Let μ^λ be the spectral measure for H_λ and ψ. This measure encodes the spectral properties of $H_\lambda \restriction \mathcal{H}_\psi$.

With the exception of Subsections 11 and 12, in this lecture we will focus on $H_\lambda \restriction \mathcal{H}_\psi$ and μ^λ. Hence, unless otherwise stated, we will always assume that $\mathcal{H} = \mathcal{H}_\psi$, namely that ψ is a cyclic vector for H_0.

The identities

$$\begin{aligned}(H_\lambda - z)^{-1} - (H_0 - z)^{-1} &= (H_\lambda - z)^{-1}(H_0 - H_\lambda)(H_0 - z)^{-1}\\ &= (H_0 - z)^{-1}(H_0 - H_\lambda)(H_\lambda - z)^{-1},\end{aligned} \tag{1}$$

yield

$$\begin{aligned}(H_\lambda - z)^{-1}\psi &= (H_0 - z)^{-1}\psi - \lambda(\psi|(H_0 - z)^{-1}\psi)(H_\lambda - z)^{-1}\psi,\\ (H_0 - z)^{-1}\psi &= (H_\lambda - z)^{-1}\psi + \lambda(\psi|(H_\lambda - z)^{-1}\psi)(H_0 - z)^{-1}\psi.\end{aligned} \tag{2}$$

Denote

$$F_\lambda(z) := (\psi|(H_\lambda - z)^{-1}\psi) = \int_{\mathbb{R}} \frac{\mathrm{d}\mu^\lambda(x)}{x-z}.$$

Note that if $z \in \mathbb{C}_+$, then $F_\lambda(z)$ is the Borel transform and $\mathrm{Im} F_\lambda(z)$ is the Poisson transform of μ^λ.

The second identity in (2) yields

$$F_0(z) = F_\lambda(z)(1 + \lambda F_0(z)),$$

and so

$$F_\lambda(z) = \frac{F_0(z)}{1+\lambda F_0(z)}. \tag{3}$$

In particular,

$$\mathrm{Im} F_\lambda(z) = \frac{\mathrm{Im} F_0(z)}{|1+\lambda F_0(z)|^2}. \tag{4}$$

These elementary identities will play a key role in our study. The function

$$G(x) := \int_{\mathbb{R}} \frac{\mathrm{d}\mu^0(t)}{(x-t)^2}$$

will also play an important role. Recall that $G(x) = \infty$ for μ^0-a.e. x.

Before reading this lecture it is a good idea that you read again Lecture 2.

2. Aronszajn-Donoghue theorem

Recall that the limit

$$F_\lambda(x) := \lim_{y \downarrow 0} F_\lambda(x + \mathrm{i}y)$$

exists and is finite and non-zero for Lebesgue a.e. x.

For $\lambda \neq 0$ define

$$\begin{aligned} S_\lambda &:= \{x \in \mathbb{R} : F_0(x) = -\lambda^{-1},\ G(x) = \infty\}, \\ T_\lambda &:= \{x \in \mathbb{R} : F_0(x) = -\lambda^{-1},\ G(x) < \infty\}, \\ L &:= \{x \in \mathbb{R} : \mathrm{Im} F_0(x) \neq 0\}. \end{aligned}$$

In words, S_λ is the set of all $x \in \mathbb{R}$ such that $\lim_{y\downarrow 0} F_0(x+\mathrm{i}y)$ exists and is equal to $-\lambda^{-1}$, etc. Any two sets in the collection $\{S_\lambda, T_\lambda, L\}_{\lambda\neq 0}$ are disjoint.

As usual, $\delta(y)$ denotes the delta-measure of $y \in \mathbb{R}$, $\delta(y)(f) = f(y)$.

Theorem 2.1. (1) T_λ is the set of eigenvalues of H_λ. Moreover,

$$\mu^\lambda_{\rm pp} = \sum_{x_n \in T_\lambda} \frac{1}{\lambda^2 G(x_n)} \delta(x_n).$$

(2) $\mu^\lambda_{\rm sc}$ is concentrated on S_λ.
(3) For all λ L is the essential support of the ac spectrum of H_λ and $\mathrm{sp}_{\rm ac}(H_\lambda) = \mathrm{sp}_{\rm ac}(H_0)$.
(4) The measures $\{\mu^\lambda_{\rm sing}\}_{\lambda \in \mathbb{R}}$ are mutually singular. In other words, if $\lambda_1 \neq \lambda_2$, then the measures $\mu^{\lambda_1}_{\rm sing}$ and $\mu^{\lambda_2}_{\rm sing}$ are concentrated on disjoint sets.

Proof. (1) The eigenvalues of H_λ are precisely the atoms of μ^λ. Let $\tilde{T}_\lambda = \{x \in \mathbb{R} : \mu^\lambda(\{x\}) \neq 0\}$. Since

$$\mu^\lambda(\{x\}) = \lim_{y \downarrow 0} y \mathrm{Im} F_\lambda(x + \mathrm{i}y) = \lim_{y \downarrow 0} \frac{y \mathrm{Im} F_0(x + \mathrm{i}y)}{|1 + \lambda F_0(x + \mathrm{i}y)|^2}, \tag{5}$$

$\tilde{T}_\lambda \subset \{x : F_0(x) = -\lambda^{-1}\}$. The relation (5) yields

$$\mu^\lambda(\{x\}) \leq \lambda^{-2} \lim_{y \downarrow 0} \frac{y}{\mathrm{Im} F_0(x + \mathrm{i}y)} = \frac{1}{\lambda^2 G(x)},$$

and so $\tilde{T}_\lambda \subset \{x : F_0(x) = -\lambda^{-1}, G(x) < \infty\} = T_\lambda$. On the other hand, if $F_0(x) = -\lambda^{-1}$ and $G(x) < \infty$, then

$$\lim_{y \downarrow 0} \frac{F_0(x + \mathrm{i}y) - F_0(x)}{\mathrm{i}y} = G(x) \tag{6}$$

(the proof of this relation is left for homework). Hence, if $x \in T_\lambda$, then

$$F_0(x + \mathrm{i}y) = \mathrm{i}yG(x) - \lambda^{-1} + o(y),$$

and

$$\mu^\lambda(\{x\}) = \lim_{y \downarrow 0} \frac{y \mathrm{Im} F_0(x + \mathrm{i}y)}{|1 + \lambda F_0(x + \mathrm{i}y)|^2} = \frac{1}{\lambda^2 G(x)} > 0.$$

Hence $T_\lambda = \tilde{T}_\lambda$, and for $x \in \tilde{T}_\lambda$, $\mu^\lambda(\{x\}) = 1/\lambda^2 G(x)$. This yields (1).
(2) By Theorem 1 in Lecture 2, $\mu^\lambda_{\rm sing}$ is concentrated on the set

$$\{x : \lim_{y \downarrow 0} \mathrm{Im} F_\lambda(x + \mathrm{i}y) = \infty\}.$$

The formula (4) yields that $\mu^\lambda_{\rm sing}$ is concentrated on the set $\{x : F_0(x) = -\lambda^{-1}\}$. If $F_0(x) = -\lambda^{-1}$ and $G(x) < \infty$, then by (1) x is an atom of μ^λ. Hence, $\mu^\lambda_{\rm sc}$ is concentrated

on the set $\{x : F_0(x) = -\lambda^{-1}, G(x) = \infty\} = S_\lambda$.
(3) By Theorem 1 in Lecture 2,

$$\mathrm{d}\mu_{\mathrm{ac}}^{\lambda}(x) = \pi^{-1}\mathrm{Im}F_\lambda(x)\mathrm{d}x.$$

On the other hand, by the formula (4), the sets $\{x : \mathrm{Im}F_0(x) > 0\}$ and $\{x : \mathrm{Im}F_\lambda(x) > 0\}$ coincide up to a set of Lebesgue measure zero. Hence, L is the essential support of the ac spectrum of H_λ for all λ. Since μ_{ac}^{0} and $\mu_{\mathrm{ac}}^{\lambda}$ are equivalent measures, $\mathrm{sp}_{\mathrm{ac}}(H_0) = \mathrm{sp}_{\mathrm{ac}}(H_\lambda)$.
(4) By (1) and (2), for $\lambda \neq 0$ $\mu_{\mathrm{sing}}^{\lambda}$ is concentrated on the set $\{x : F_0(x) = -\lambda^{-1}\}$. By Theorem 1 in Lecture 2, μ_{sing}^{0} is supported on $\{x : \mathrm{Im}F_0(x) = \infty\}$. This yields the statement. □

Digression: In the Lecture 6 we have used the following fact:

Theorem 2.2. Let A be a selfadjoint operator on a Hilbert space $\mathcal{H}$ and $\{\delta_j\}_{1\le j\le n}$ a finite collection of vectors in $\mathcal{H}$. Let

$$A_{\lambda_1,\cdots,\lambda_n} = A + \sum_{j=1}^{n} \lambda_j(\delta_j|\cdot)\delta_j.$$

Then for all $\lambda_1, \cdots, \lambda_n$, $\mathrm{sp}_{\mathrm{ac}}(A_{\lambda_1,\cdots,\lambda_n}) = \mathrm{sp}_{\mathrm{ac}}(A)$ and $\Sigma_{\mathrm{ac}}^{\mathrm{ess}}(A_{\lambda_1,\cdots,\lambda_n}) = \Sigma_{\mathrm{ac}}^{\mathrm{ess}}(A)$.

Theorem 2.2. follows from Theorem 2.1 (3) if $n = 1$. By induction, the statement holds for all $n > 0$.

3. The rank one Wegner estimate.

Definition 3.1. Let $0 < \tau \le 1$. A Borel probability measure P on $\mathbb{R}$ is called τ-regular if there exists finite constant M_τ such that for $x \in \mathbb{R}$ and $r > 0$,

$$P([x - r, x + r]) \le M_\tau r^\tau.$$

A τ-regular measure may not have density. If $\mathrm{d}P(\lambda) = f(\lambda)\mathrm{d}\lambda$ and $f \in L^p(\mathbb{R})$ for some $p \in (1, \infty]$, then P is $1 - p^{-1}$–regular.

Theorem 3.2. Let P be a τ-regular probability measure on $\mathbb{R}$ with $\tau \le 1$. Then for $0 \le s < \tau$,

$$\sup_{z\in\mathbb{C}\setminus\mathbb{R}} \mathbb{E}\left|(\psi|(H_\lambda - z)^{-1}\psi)\right|^s \le \frac{\tau}{\tau - s} M_\tau^{s/\tau}.$$

Proof. It follows from the formula (3) that

$$\sup_{z\in\mathbb{C}\setminus\mathbb{R}} \mathbb{E}\left|(\psi|(H_\lambda - z)^{-1}\psi)\right|^s = \sup_{z\in\mathbb{C}\setminus\mathbb{R}} \int_{\mathbb{R}} \left|(\psi|(H_\lambda - z)^{-1}\psi)\right|^s \mathrm{d}P(\lambda)$$
$$\leq \sup_{x\in\mathbb{R}} \int_{\mathbb{R}} |x-\lambda|^{-s}\mathrm{d}P(\lambda).$$

Let $\delta > 0$. Then for all x,

$$\int_{\mathbb{R}} |x-\lambda|^{-s}\mathrm{d}P(\lambda) \leq \int_{|x-\lambda|^{-s}\leq\delta} |x-\lambda|^{-s}\mathrm{d}P(\lambda) + \int_{|x-\lambda|^{-s}\geq\delta} |x-\lambda|^{-s}\mathrm{d}P(\lambda)$$
$$\leq \delta + \int_\delta^\infty P(\{|x-\lambda|^{-s}\geq t\})\mathrm{d}t$$
$$\leq \delta + M_\tau \int_\delta^\infty t^{-\tau/s}\mathrm{d}t$$
$$\leq \delta + \frac{sM_\tau \delta^{1-\tau/s}}{\tau - s}$$

(we have used homework problem 2 in Lecture 1). Optimizing over δ we derive the statement. □

The above proof is based on [AG]. The trivial example $H_0\psi = 0$ shows that Theorem 3.2 in general fails if $s = 1$.

The Wegner type estimates play a very important role in study of Anderson type Hamiltonians (Theorem 3.2 is the simplest Wegner type estimate). Note that the function $\mathbb{C}\setminus\mathbb{R} \ni z \to |(\psi|(H_\lambda - z)^{-1}\psi)|^s$ is certainly unbounded if, for example, H_λ has eigenvalues. On the other hand, its average over a general class of probability measures is bounded.

4. Spectral averaging

For every Borel set $B \subset \mathbb{R}$ set

$$\overline{\mu}(B) = \int_{\mathbb{R}} \mu^\lambda(B)\mathrm{d}\lambda.$$

Obviously, $\overline{\mu}$ is a Borel measure on $\mathbb{R}$. The following (somewhat surprising) result is often called *spectral averaging* :

Theorem 4.1. The measure $\overline{\mu}$ is equal to the Lebesgue measure and for all $f \in L^1(\mathbb{R}, \mathrm{d}x)$,

$$\int_{\mathbb{R}} f(x)\mathrm{d}x = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(x)\mathrm{d}\mu^\lambda(x)\right] \mathrm{d}\lambda.$$

Proof. For any positive Borel function f,

$$\int_{\mathbb{R}} f(t)\mathrm{d}\overline{\mu}(t) = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(t)\mathrm{d}\mu^\lambda(t)\right] \mathrm{d}\lambda$$

(both sides are allowed to be infinity). Let

$$f(t) = \frac{y}{(t-x)^2 + y^2},$$

where $y > 0$. Then

$$\int_{\mathbb{R}} f(t)\mathrm{d}\mu^\lambda(t) = \mathrm{Im}F_\lambda(x+\mathrm{i}y) = \frac{\mathrm{Im}F_0(x+\mathrm{i}y)}{|1+\lambda F_0(x+\mathrm{i}y)|^2}.$$

By the residue calculus,

$$\int_{\mathbb{R}} \frac{\mathrm{Im}F_0(x+\mathrm{i}y)}{|1+\lambda F_0(x+\mathrm{i}y)|^2}\mathrm{d}\lambda = \pi, \tag{7}$$

and so the Poisson transform of $\overline{\mu}$ exists and is identicaly equal to π, the Poisson transform of the Lebesgue measure. By Theorem 3 in Lecture 2, $\overline{\mu}$ is equal to the Lebesgue measure. □

For latter applications, we mention the following obvious consequence of Theorem 4.1:

Corollary 4.2. Let P be an absolutely continuous Borel probability measure on $\mathbb{R}$. Then the averaged measure

$$\nu(\,\cdot\,) = \int_{\mathbb{R}} \mu^\lambda(\,\cdot\,)\mathrm{d}P(\lambda)$$

is absolutely continuous w.r.t. the Lebesgue measure.

Proof. If B is a Borel set of Lebesgue measure zero, then by Theorem 4.1, $\mu^\lambda(B) = 0$ for Lebesgue a.e. λ. Since P has a density, $\nu(B) = 0$. □

The spectral averaging is a mathematical gem which has been rediscovered by many authors. A detailed list of references can be found in [Si2].

5. Simon-Wolff theorems

Theorem 5.1 Let $B \subset \mathbb{R}$ be a Borel set. Then the following statements are equivalent:
(1) $G(x) < \infty$ for Lebesgue a.e. $x \in B$.
(2) $\mu^\lambda_{\mathrm{cont}}(B) = 0$ for Lebesgue a.e. λ.

In particular, the spectrum of H_λ in an interval (a,b) is pure point for Lebesgue a.e. λ iff $G(x)<\infty$ for Lebesgue a.e. x.

Proof. (1) $\Rightarrow$ (2). If $G(x)<\infty$ for Lebesgue a.e. $x\in B$, then $\mathrm{Im}F_0(x)=0$ for Lebesgue a.e. $x\in B$. Hence, for all λ, $\mathrm{Im}F_\lambda(x)=0$ for Lebesgue a.e. $x\in B$, and

$$\mu^\lambda_{\mathrm{ac}}(B)=\pi^{-1}\int_B \mathrm{Im}F_\lambda(x)\mathrm{d}x=0.$$

By Theorem 1.1, the measure $\mu^\lambda_{\mathrm{sc}}\restriction B$ is concentrated on the set $A=\{x\in B: G(x)=\infty\}$. Since A has Lebesgue measure zero, by the spectral averaging

$$\int_{\mathbb{R}}\mu^\lambda_{\mathrm{sc}}(A)\mathrm{d}\lambda\le\int_{\mathbb{R}}\mu^\lambda(A)\mathrm{d}\lambda=|A|=0.$$

Hence, $\mu^\lambda_{\mathrm{sc}}(A)=0$ for Lebesgue a.e. $\lambda\in\mathbb{R}$, and so $\mu^\lambda_{\mathrm{sc}}(B)=0$ for Lebesgue a.e. λ.
(2) $\Rightarrow$ (1). Assume that the set $A=\{x\in B:G(x)=\infty\}$ has positive Lebesgue measure. By Theorem 1, $\mu^\lambda_{\mathrm{pp}}(A)=0$ for all $\lambda\neq 0$. By the spectral averaging,

$$\int_{\mathbb{R}}\mu^\lambda_{\mathrm{cont}}(A)\mathrm{d}\lambda=\int_{\mathbb{R}}\mu^\lambda(A)\mathrm{d}\lambda=|A|>0.$$

Hence, for a set of λ of positive Lebesgue measure, $\mu^\lambda_{\mathrm{cont}}(A)>0$. □

Recall that the essential spectrum of H_λ is independent of λ.

Theorem 5.2. Let $B\subset \mathrm{sp}_{\mathrm{ess}}(H_0)$ be a Borel set. Then the following statements are equivalent:
(1) $\mathrm{Im}F_0(x)>0$ for Lebesgue a.e. $x\in B$.
(2) $\mu^\lambda_{\mathrm{sing}}(B)=0$ for Lebesgue a.e. λ.
In particular, if $(a,b)\subset\mathrm{sp}_{\mathrm{ess}}(H_0)$, then the spectrum of H_λ in (a,b) is purely a.c. for Lebesgue a.e. λ iff $\mathrm{Im}F_0(x)>0$ for Lebesgue a.e. $x\in(a,b)$.

Proof. (1) $\Rightarrow$ (2). By Theorem (1), for $\lambda\neq 0$ the measure $\mu^\lambda_{\mathrm{sing}}\restriction B$ is concentrated on the set $A=\{x\in B:\mathrm{Im}F_0(x)=0\}$. Since A has Lebesgue measure zero, by the spectral averaging,

$$\int_{\mathbb{R}}\mu^\lambda_{\mathrm{sing}}(A)\mathrm{d}\lambda\le\int_{\mathbb{R}}\mu^\lambda(A)\mathrm{d}\lambda=0.$$

Hence, for Lebesgue a.e. λ, $\mu^\lambda_{\mathrm{sing}}(A)=0$.
(2) $\Rightarrow$ (1). Assume that the set $A=\{x\in B:\mathrm{Im}F_0(x)=0\}$ has positive Lebesgue measure. Clearly, $\mu^\lambda_{\mathrm{ac}}(A)=0$ for all λ, and by the spectral averaging,

$$\int_{\mathbb{R}}\mu^\lambda_{\mathrm{sing}}(A)\mathrm{d}\lambda=\int_{\mathbb{R}}\mu^\lambda(A)\mathrm{d}\lambda=|A|>0.$$

Hence, for a set of λ of positive Lebesgue measure, $\mu^\lambda_{\rm sing}(A) > 0$. $\Box$

Theorem 5.3. Let $B \subset {\rm sp}_{\rm ess}(H_0)$ be a Borel set. Then the following statements are equivalent:
(1) ${\rm Im}F_0(x) = 0$ and $G(x) = \infty$ for Lebesgue a.e. $x \in B$.
(2) $\mu^\lambda_{\rm ac}(B) + \mu^\lambda_{\rm pp}(B) = 0$ for Lebesgue a.e. λ.
In particular, if $(a,b) \subset {\rm sp}_{\rm ess}(H_0)$, then the spectrum of H_λ in (a,b) is purely singular continuous for Lebesgue a.e. λ iff ${\rm Im}F_0(x) = 0$ and $G(x) = \infty$ for Lebesgue a.e. x.

The proof of Theorem 5.3 is left for homework.

Theorem 5.1. is the well-known result of Simon-Wolff [SW]. Although Theorems 5.2 and 5.3 are well-known to the workers in the field, I am not aware of the convenient reference.

6. Some remarks on spectral instability

The absolutely continuous spectrum is stable under rank one perturbations: if $(a,b) \subset {\rm sp}_{\rm ac}(H_0)$, then for all λ, $(a,b) \subset {\rm sp}_{\rm ac}(H_\lambda)$. The situation is more complicated in the case of the singular continuous spectrum. There are examples where sc spectrum is stable, namely H_λ has purely singular continuous spectrum in (a,b) for all $\lambda \in \mathbb{R}$. There are also examples where H_0 has purely sc spectrum in (a,b), but H_λ has pure point spectrum for all $\lambda \neq 0$. These examples are discussed in your project.

A. Gordon [Gor] and del Rio-Makarov-Simon [DMS] have proven that pp spectrum is *always* unstable for generic λ. More precisely, we have

Theorem 6.1. The set

$$\{\lambda \,:\, H_\lambda \text{ has no eigenvalues in } {\rm sp}(H_0)\}$$

is dense G_δ in $\mathbb{R}$.

The proof of this theorem will be included in the final version of the notes.

Assume that $(a,b) \subset {\rm sp}(H_0)$ and that for Lebesgue a.e. $x \in (a,b)$, $G(x) < \infty$. Then the spectrum of H_λ in (a,b) is pure point for Lebesgue a.e. λ. However, by Theorem 6.1, there is a dense G_δ set of λ's such that H_λ has purely singular continuous spectrum in (a,b) (of course, H_λ has no ac spectrum in (a,b) for all λ).

7. The Aizenman characterization

By the spectral theorem, for all λ there exists a unitary map $U_\lambda : \mathcal{H} \to L^2(\mathbb{R}, {\rm d}\mu^\lambda)$ such that $U_\lambda H_\lambda U_\lambda^{-1}$ is the operator of multiplication by x and $U_\lambda \psi = 1$. The Aizenman characterization [Aiz] describes the action of U_λ on spectral intervals where H_λ has pure point spectrum for Lebesgue a.e. λ. Our exposition of the Aizenman characterization follows closely [DJLS].

Let (a, b) be a finite interval. By the Simon-Wolff theorem, the spectrum of H_λ in (a, b) is pure point iff $G(x) < \infty$ for Lebesgue a.e. $x \in (a, b)$.

Theorem 7.1 Let $\mathcal{G} := \{x \in (a, b) : G(x) < \infty, F_0(x) \neq 0\}$ and assume that $(a, b) \setminus \mathcal{G}$ has Lebesgue measure zero. Then:
(1) For all $x \in \mathcal{G}$ the limit

$$\psi_x := \lim_{y \downarrow 0} (H_0 - x - \mathrm{i}y)^{-1} \psi$$

exists.
(2) Let $\lambda \neq 0$ be such that $\mu^\lambda \restriction (a, b)$ is concentrated on $\mathcal{G}$. Then for all $\varphi \in \mathcal{H}$ and $x \in \mathcal{G}$,

$$(U_\lambda \varphi)(x) = -\lambda (\psi_x | \varphi).$$

Remark: In the next subsection we will use the notation $\psi_x = (H_0 - x - \mathrm{i}0)^{-1} \psi$.
Proof. We start with the identity

$$\frac{(H_\lambda - z)^{-1} \psi}{(\psi | (H_\lambda - z)^{-1} \psi)} = \frac{(H_0 - z)^{-1} \psi}{(\psi | (H_0 - z)^{-1} \psi)} \tag{8}$$

which holds for all $z \in \mathbb{C} \setminus \mathbb{R}$ and $\lambda \neq 0$. This identity follows from the relations (2). Let $x \in \mathcal{G}$. Then $F_0(x)$ exists and is equal to $-\lambda^{-1}$ for some $\lambda \neq 0$. Substitute this value of λ in (8). By Theorem 1, x is an eigenvalue of H_λ. Since

$$\lim_{y \downarrow 0} -\mathrm{i}y (H_\lambda - x - \mathrm{i}y)^{-1} = (\tilde{\psi}_x | \cdot) \tilde{\psi}_x,$$

where $\tilde{\psi}_x$ is the normalized eigenfunction of H_λ for x, we have

$$\lim_{y \downarrow 0} \frac{(H_\lambda - x - \mathrm{i}y)^{-1} \psi}{(\psi | (H_\lambda - x - \mathrm{i}y)^{-1} \psi)} = \lim_{y \downarrow 0} \frac{-\mathrm{i}y (H_\lambda - x - \mathrm{i}y)^{-1} \psi}{-\mathrm{i}y (\psi | (H_\lambda - x - \mathrm{i}y)^{-1} \psi)} = \frac{(\tilde{\psi}_x | \psi)}{|(\tilde{\psi}_x | \psi)|^2} \tilde{\psi}_x. \tag{9}$$

The relations (8), (9) and $F_0(x) = -\lambda^{-1}$ yield

$$\lim_{y \downarrow 0} (H_0 - x - \mathrm{i}y)^{-1} \psi = -\frac{(\tilde{\psi}_x | \psi)}{\lambda |(\tilde{\psi}_x | \psi)|^2} \tilde{\psi}_x.$$

Setting

$$\psi_x := -\frac{(\tilde{\psi}_x | \psi)}{\lambda |(\tilde{\psi}_x | \psi)|^2} \tilde{\psi}_x$$

we derive (1).

To prove (2), observe that for $x \in \mathcal{G}$, $(U_\lambda \varphi)(x)$ is the inner product of φ and the eigenfunction of H_λ for x normalized as $(\psi|\cdot) = 1$. (*It is very important that you understand this point—this is indeed a test whether you have understood the statement of the spectral theorem*). This eigenfunction is obviously $-\lambda\psi_x$ and (2) follows. □

8. The Aizenmann estimate

This subsection is a continuation of Subsection 7. Let (a,b) again be an interval such that $G(x) < \infty$ for Lebesgue a.e. $x \in (a,b)$. Let P be an absolutely continuous probability measure on $\mathbb{R}$, $\mathrm{d}P(\lambda) = p(\lambda)\mathrm{d}\lambda$.

Theorem 8.1. Assume that $p \in L^\infty(\mathbb{R})$ and that p has compact support. Let $\varphi \in \mathcal{H}$, $\|\varphi\| = 1$. Then for all $0 < s < 1$,

$$\mathbb{E}\sup_t \left|(\psi|\mathbf{1}_{[a,b]}(H_\lambda)\mathrm{e}^{-\mathrm{i}tH_\lambda}\varphi)\right| \le C_s \left(\int_a^b \mathbb{E}|(\varphi|(H_\lambda - x - \mathrm{i}0)^{-1}\psi)|^s \mathrm{d}x\right)^{1/(2-s)} \tag{10}$$

where $C_s := D^{s/(2-s)}\|p\|_\infty^{1/(2-s)}$ and $D := \mathrm{diam}(\mathrm{supp}\ p) = \max\{|\lambda - \lambda'| \ :\ \lambda, \lambda' \in \mathrm{supp}\, p\}$.

Theorem 8.1 was proven by Aizenman [Aiz]. The proof below follows closely [DJLS]. At the end of this subsection we discuss some technical extensions of Theorem 8.1.

Aizenman [Aiz] has used Theorem 8.1 to give the first proof of dynamical localization for Anderson type Hamiltonians (including the Anderson model on $\mathbb{Z}^d$). Although it is possible to prove dynamical localization without use of Theorem 8.1 (see the next subsection and Lecture 10), the original approach of Aizenman is very instructive.

Proof of Theorem 8.1. It suffices to prove that for all $\lambda_0 \in \mathrm{supp}\, p$,

$$\mathbb{E}\sup_t \left|(\psi|\mathbf{1}_{[a,b]}(H_\lambda)\mathrm{e}^{-\mathrm{i}tH_\lambda}\varphi)\right| \le C_s \left(\int_a^b |(\psi|(H_{\lambda_0} - x - \mathrm{i}0)^{-1}\varphi)|^s \mathrm{d}x\right)^{1/(2-s)}. \tag{11}$$

Taking the expectation of (11) w.r.t. λ_0 and using the Jensen inequality (note that $0 < 1/(2-s) < 1$), one derives (10) from (11).

We proceed to prove (11). By the spectral theorem,

$$(\psi|\mathbf{1}_{[a,b]}(H_\lambda)\mathrm{e}^{-\mathrm{i}tH_\lambda}\varphi) = \int_a^b \mathrm{e}^{-\mathrm{i}tx}(U_\lambda\varphi)(x)\mathrm{d}\mu^\lambda(x),$$

where U_λ and μ^λ are as in the previous subsection. Note that

$$\int_{\mathbb{R}} |(U_\lambda\varphi)(x)|^2 \mathrm{d}\mu^\lambda(x) = \|\varphi\|^2 = 1.$$

Since

$$\int |g|\mathrm{d}\mu^\lambda \le \left(\int |g|^2\mathrm{d}\mu^\lambda\right)^{(1-s)/(2-s)} \left(\int |g|^s\mathrm{d}\mu^\lambda\right)^{1/(2-s)},$$

(write $|g| = |g|^{2(1-s)/(2-s)}|g|^{s/(2-s)}$ and use the Hölder inequality with $p = (2-s)/(1-s)$, $q = 2-s$), we derive

$$\sup_t \left|(\psi|\mathbf{1}_{[a,b]}(H_\lambda)\mathrm{e}^{-\mathrm{i}tH_\lambda}\varphi)\right| \le \int_a^b |U_\lambda\varphi(x)|\mathrm{d}\mu^\lambda(x) \le \left(\int_a^b |(U_\lambda\varphi)(x)|^s\mathrm{d}\mu^\lambda(x)\right)^{1/(2-s)}. \tag{12}$$

Now we think of H_λ as rank-one perturbation of H_{λ_0},

$$H_\lambda = H_{\lambda_0} + (\lambda-\lambda_0)(\psi|\cdot)\psi.$$

Theorem 8.1 yields that for Lebesgue a.e. λ and for μ^λ-a.e. $x \in (a,b)$,

$$(U_\lambda\varphi)(x) = -(\lambda-\lambda_0)((H_{\lambda_0} - x - \mathrm{i}0)^{-1}\psi|\varphi).$$

Substituting in (12) we derive

$$\sup_t \left|(\psi|\mathbf{1}_{[a,b]}(H_\lambda)\mathrm{e}^{-\mathrm{i}tH_\lambda}\varphi)\right| \le D^{s/(2-s)} \left(\int_a^b |(\varphi|H_{\lambda_0} - x - \mathrm{i}0)^{-1}\psi)|^s\mathrm{d}\mu^\lambda(x)\right)^{1/(2-s)} \tag{13}$$

The Jensen inequality and the spectral averaging yield

$$\mathbb{E}\left(\int_a^b |(\varphi|(H_{\lambda_0} - x - \mathrm{i}0)^{-1}\psi)|^s\mathrm{d}\mu^\lambda(x)\right)^{1/(2-s)}$$
$$\le \left(\mathbb{E}\int_a^b |(\varphi|(H_{\lambda_0} - x - \mathrm{i}0)^{-1}\psi)|^s\mathrm{d}\mu^\lambda(x)\right)^{1/(2-s)}$$

$$\mathbb{E}\int_a^b |(\varphi|(H_{\lambda_0} - x - \mathrm{i}0)^{-1}\psi)|^s\mathrm{d}\mu^\lambda(x) = \int_a^b\int_{\mathbb{R}} |(\varphi|(H_{\lambda_0} - x - \mathrm{i}0)^{-1}\psi)|^s p(\lambda)\mathrm{d}\mu^\lambda(x)\mathrm{d}\lambda$$
$$\le \|p\|_\infty \int_a^b |(\varphi|(H_{\lambda_0} - x - \mathrm{i}0)^{-1}\psi)|^s\mathrm{d}x.$$

□

The condition on density p can be relaxed:

Theorem 8.2 Assume that $p \in L^\infty(\mathbb{R})$ and that $\mathbb{E}(|\lambda|^r) < \infty$ for some $r > 0$. Then

$$\mathbb{E}\sup_t \left|(\psi|\mathbf{1}_{[a,b]}(H_\lambda)\mathrm{e}^{-\mathrm{i}tH_\lambda}\varphi)\right| \leq C\left(\int_a^b \mathbb{E}\,|(\psi|(H_\lambda - x - \mathrm{i}0)^{-1}\varphi)|^s \mathrm{d}x\right)^{r/(s+r(2-s))} \tag{14}$$

where

$$C = \mathbb{E}(|\lambda|^r)^{s/(s+r(2-s))}\|p\|_\infty^{r/(s+r(2-s))}C(r,s)$$

and

$$C(r,s) = 2^{sr/(s+r(2-s))}((2-s)r/s)^{s/(s+r(2-s))} + (s/((2-s)r))^{r(2-s)/(s+r(2-s))}.$$

Proof. For $\delta > 0$ write $\mathbb{E}_{\underline{\delta}}(X) = \int_{-\delta}^{\delta} X\mathrm{d}P$, $\mathbb{E}_{\overline{\delta}}(X) = \mathbb{E}(X) - \mathbb{E}_{\underline{\delta}}(X)$. The proof of Theorem 8.1 yields

$$\begin{aligned}&\mathbb{E}_{\underline{\delta}}\sup_t |(\psi|\mathbf{1}_{[a,b]}(H_\lambda)\mathrm{e}^{-\mathrm{i}tH_\lambda}\varphi)| \\ &\qquad\leq (2\delta)^{s/(2-s)}\|p\|_\infty^{1/(2-s)}\left(\int_a^b \mathbb{E}\,|(\psi|(H_\lambda - x - \mathrm{i}0)^{-1}\varphi)|^s \mathrm{d}x\right)^{1/(2-s)}\end{aligned}$$

On the other hand, since

$$\sup_t |(\psi|\mathbf{1}_{[a,b]}(H_\lambda)\mathrm{e}^{-\mathrm{i}tH_\lambda}\varphi)| \leq 1,$$

$$\mathbb{E}_{\overline{\delta}}\sup_t |(\psi|\mathbf{1}_{[a,b]}(H_\lambda)\mathrm{e}^{-\mathrm{i}tH_\lambda}\varphi)| \leq \delta^{-r}\mathbb{E}(|\lambda|^r).$$

Since $\mathbb{E} = \mathbb{E}_{\underline{\delta}} + \mathbb{E}_{\overline{\delta}}$, optimizing over δ we derive the statement. □

The condition $p \in L^\infty(\mathbb{R})$ can be replaced with $p \in L^q(\mathbb{R}, \mathrm{d}\lambda)$ for some $q > 1$. This extension is derived by taking first the expectation over $\{\lambda : p(\lambda) \leq \delta\}$ and using Theorem 8.2. The expectation over $\{\lambda : p(\lambda) \geq \delta\}$ is bounded by $\delta^{1-q}\int_{\mathbb{R}} p(\lambda)^q \mathrm{d}\lambda$. After that one optimizes over δ.

Theorem 8.3 Assume that $p \in L^q(\mathbb{R})$ for some $q > 1$ and that $\mathbb{E}(|\lambda|^r) < \infty$ for some $r > 0$. Then

$$\mathbb{E}\sup_t \left|(\psi|\mathbf{1}_{[a,b]}(H_\lambda)\mathrm{e}^{-\mathrm{i}tH_\lambda}\varphi)\right| \leq C\left(\int_a^b \mathbb{E}\,|(\psi|(H_\lambda - x - \mathrm{i}0)^{-1}\varphi)|^s \mathrm{d}x\right)^{(q-1)r/((q-1)s+qr(2-s))} \tag{15}$$

where

$$C = \|p\|_q^{rq/((q-1)s+qr(2-s))} (\mathbb{E}(|\lambda|^r))^{s(q-1)/((q-1)s+qr(2-s))} C_1(r,s,q)$$

$$\begin{aligned} C_1(r,s,q)) &= C(r,s)^{(q-1)(s+r(2-s))/((q-1)s+qr(2-s))} \\ &\times (((q-1)(s+2(r-s))/r)^{r/((q-1)s+qr(2-s))} \\ &\qquad + (r/((q-1)(s+2(r-s)))^{(q-1)(s+r(2-s))/((q-1)s+qr(2-s))}), \end{aligned}$$

and $C_1(r,s)$ is as in Theorem 8.2.

9. Boole's equality

So far we have used the rank one perturbation theory and harmonic analysis to study spectral theory. In this and the next subsection we will turn things around and use rank one perturbation theory and spectral theory to reprove some well known results in harmonic analysis. This subsection deals with Boole's equality/inequality and is based on [DJLS] and [Po2].

Let ν be a finite positive Borel measure on $\mathbb{R}$ and $F_\nu(z)$ its Borel transform. As usual, we denote

$$F_\nu(x) := \lim_{y\downarrow 0} F_\nu(x+\mathrm{i}y).$$

The following result is known as Boole's equality:

Proposition 9.1. Assume that ν is a pure point measure with finitely many atoms. Then

$$|\{x \,:\, F_\nu(x) > t\}| = |\{x \,:\, F_\nu(x) < -t\}| = \frac{\nu(\mathbb{R})}{t}.$$

Proof. We will prove that $|\{x \,:\, F_\nu(x) > t\}| = \nu(\mathbb{R})/t$. Let $\{x_j\}_{1\leq j\leq n}$ be the support of ν and $\alpha_j = \nu(\{x_j\})$ the atoms of ν. W.l.o.g. we may assume that $\nu(\mathbb{R}) = \sum_j \alpha_j = 1$. Clearly,

$$F_\nu(x) = \sum_{j=1}^{n} \frac{\alpha_j}{x_j - x}.$$

Set $x_0 = -\infty$, $x_{n+1} = \infty$. Since $F_\nu'(x) > 0$ for $x \neq x_j$, the function $F_\nu(x)$ is strictly increasing on (x_j, x_{j+1}), with vertical asymptots at $x_j, 1 \leq j \leq n$. Let $r_1, \cdots, r_n$ be the solution of the equation $F_\nu(x) = t$, ordered in the increasing order. Then

$$|\{x \,:\, F_\nu(x) > t\}| = \sum_{j=1}^{n} (x_j - r_j).$$

On the other hand, the equation $F_\nu(x) = t$ is equivalent to

$$\sum_{k=1}^n \alpha_k \prod_{j\neq k}(x_j - x) = t \prod_{j=1}^n (x_j - x),$$

or

$$\prod_{j=1}^n (x_j - x) - t^{-1} \sum_{k=1}^n \alpha_k \prod_{j\neq k}(x_j - x) = 0.$$

Since $\{r_j\}$ are all the roots of the polynomial on the r.h.s.,

$$\sum_{j=1}^n r_j = t^{-1} + \sum_{j=1}^n x_j$$

and this yields the statement. $\square$

Proposition 9.1 was first by proven by G. Boole in 1867. I have learned the above elementary proof from Stas Molchanov (he atributes it to Kolmogorov). The Boole equality is another gem that has been rediscovered by many authors (see [Po2] for the references—to my knowledge the latest rediscovery is due to E. Lieb in early 90's).

The rank one perturbation theory allows for a simple proof of the optimal version of the Boole equality.

Theorem 9.2. Assume that ν is a purely singular measure. Then for all $t > 0$

$$|\{x \,:\, F_\nu(x) > t\}| = |\{x \,:\, F_\nu(x) < -t\}| = \frac{\nu(\mathbb{R})}{t}.$$

Proof. W.l.o.g. we may assume that $\nu(\mathbb{R}) = 1$. Let H_0 be the operator of multiplication by x on $L^2(\mathbb{R}, \mathrm{d}\nu)$ and $\psi \equiv 1$. Let $H_\lambda = H_0 + \lambda(\psi|\cdot)\psi$ and let μ^λ be the spectral measure for H_λ and ψ. Obviously, $\mu^0 = \nu$ and $F_0 = F_\nu$. Since ν is a singular measure, μ^λ is singular for all $\lambda \in \mathbb{R}$.

By Theorem 1.1, for $\lambda \neq 0$, the measure μ^λ is concentrated on the set $\{x : F_0(x) = -\lambda^{-1}\}$. Let

$$\Gamma_t = \{x \,:\, F_0(x) > t\}.$$

Then for $\lambda \neq 0$,

$$\mu^\lambda(\Gamma_t) = \begin{cases} 1 & \text{if } -t^{-1} < \lambda < 0, \\ 0 & \text{if } \lambda \leq -t^{-1} \text{ or } \lambda > 0. \end{cases}$$

By the spectral averaging,

$$|\Gamma_t| = \int_{\mathbb{R}} \mu^\lambda(\Gamma_t) = t^{-1}.$$

Similar argument yields that $|\{x : F_\nu(x) < -t\}| = t^{-1}$. $\square$

The Boole inequality fails if ν is not a singular measure. However, in general we have

Theorem 9.3. Let ν be a finite positive Borel measure on $\mathbb{R}$. Then

$$\lim_{t\to\infty} t\,|\{x : |F_\nu(x)| > t\}| = 2\nu_{\rm sing}(\mathbb{R}).$$

Theorem 9.3 is due to Vinogradov-Hruschev. Its proof (and many additional information) can be found in the paper of Poltoratskii [Po2].

10. Poltoratskii's theorem

Let us first recall the statement of the Poltoratskii theorem (Theorem 11 in Lecture 2).

Theorem 10.1. Let ν be a complex and μ a positive regular Borel measure. Let $\nu = f\mu + \nu_{\rm s}$ be the Radon-Nikodym decomposition. Let $\mu_{\rm sing}$ be the part of μ singular with respect to the Lebesgue measure. Then

$$\lim_{y\downarrow 0} \frac{F_\nu(x+{\rm i}y)}{F_\mu(x+{\rm i}y)} = f(x) \qquad \text{for } \mu_{\rm sing} - \, a.e.\, x. \tag{16}$$

The proofs in Subsections 12 depend crucially on Theorem 10.1. In this subsection we describe a new proof of Theorem 10.1 obtained recently in [JL1].

Proof. We first consider the case $\nu_{\rm s} = 0$, μ compactly supported, $f \in L^2(\mathbb{R}, {\rm d}\mu)$ real valued. W.l.o.g. we may assume that $\mu(\mathbb{R}) = 1$.

Our proof is based on the rank one perturbation theory and uses some ideas of Poltoratskii (see in particular the proof of Theorem 1.4 in [Po1]).

Consider the Hilbert space $L^2(\mathbb{R}, {\rm d}\mu)$ and let H_0 be the operator of multiplication of x. With a slight abuse of the notation we denote by $\mathbf{1}$ the constant function equal to 1 on $\mathbb{R}$. Note that

$$F_\mu(z) = (\,\mathbf{1}\,|(H_0 - z)^{-1}\mathbf{1}\,), \qquad F_{f\mu}(z) = (\,\mathbf{1}\,|(H_0 - z)^{-1} f\,).$$

For $\lambda \in \mathbb{R}$, let

$$H_\lambda = H_0 + \lambda(\,\mathbf{1}\,|\,\cdot\,)\mathbf{1},$$

and let μ^λ be the spectral measure for H_λ and $\mathbf{1}$. To simplify the notation, we write

$$F_\lambda(z) = (\,\mathbf{1}\,|(H_\lambda - z)^{-1}\mathbf{1}\,) = F_{\mu^\lambda}(z).$$

Note that with this notation, $F_0 = F_\mu$!

Recall that the measures $\{\mu^\lambda_{\rm sing}\}_{\lambda\in\mathbb{R}}$ are mutually singular (Theorem 1 in this lecture) and that the measure $\mu_{\rm sing} = \mu^0_{\rm sing}$ is supported on the set

$$\{x \in \mathbb{R} : \lim_{y\downarrow 0} {\rm Im} F_0(x + {\rm i}y) = \infty\}$$

(Theorem 1 in Lecture 2). We also recall the identity

$$F_\lambda(z) = \frac{F_0(z)}{1 + \lambda F_0(z)}. \tag{17}$$

By the spectral theorem, there exists a map

$$U_\lambda : L^2(\mathbb{R}, {\rm d}\mu) \mapsto L^2(\mathbb{R}, {\rm d}\mu^\lambda)$$

such that $U_\lambda\mathbf{1} = \mathbf{1}$ and $U_\lambda H_\lambda U_\lambda^{-1}$ is the operator of multiplication by x on $L^2(\mathbb{R}, {\rm d}\mu^\lambda)$. Hence

$$(\,\mathbf{1}\,|(H_\lambda - z)^{-1}f) = \int_{\mathbb{R}} \frac{(U_\lambda f)(x)}{x - z}{\rm d}\mu^\lambda(x) = F_{(U_\lambda f)\mu^\lambda}(z).$$

In what follows we set $\lambda = 1$ and write $U = U_1$.

Since

$$Ux^n = UH_0^n\mathbf{1} = UH_0U^{-1}Ux^{n-1} = xUx^{n-1} - (\mathbf{1}|x^{n-1})\mathbf{1},$$

we see that U takes polynomials with real coefficients to the polynomials with real coefficients. Hence, U takes real valued functions to real valued functions. In particular, Uf is a real valued function.

The identity

$$(\,\mathbf{1}\,|(H_0 - z)^{-1}f) = (1 + (\,\mathbf{1}\,|(H_0 - z)^{-1}\mathbf{1}\,))(\,\mathbf{1}\,|(H_1 - z)^{-1}f).$$

can be rewritten as

$$(\,\mathbf{1}\,|(H_0 - z)^{-1}f) = (1 + F_0(z))F_{(Uf)\mu^1}(z). \tag{18}$$

It follows that

$$\frac{{\rm Im}(\,\mathbf{1}\,|(H_0 - z)^{-1}f)}{{\rm Im}F_0(z)} = {\rm Re}F_{(Uf)\mu^1}(z) + L(z), \tag{19}$$

where

$$L(z)=\frac{\mathrm{Re}(1+F_0(z))}{\mathrm{Im}F_0(z)}\mathrm{Im}F_{(Uf)\mu^1}(z).$$

We proceed to prove the following two relations:

$$\lim_{y\downarrow 0}\mathrm{Im}F_{(Uf)\mu_1}(x+\mathrm{i}y)=0 \qquad \text{for}\ \ \mu_{\mathrm{sing}}-a.e.\ x, \tag{20}$$

$$\lim_{y\downarrow 0}L(x+\mathrm{i}y)=0 \qquad \text{for}\ \ \mu_{\mathrm{sing}}-a.e.\ x. \tag{21}$$

Let us prove (20). Using first that Uf is real-valued and then the Cauchy-Schwarz inequality, we derive

$$\mathrm{Im}F_{(Uf)\mu^1}(x+\mathrm{i}y)=P_{(Uf)\mu^1}(x+\mathrm{i}y)\leq\sqrt{P_{\mu^1}(x+\mathrm{i}y)}\sqrt{P_{(Uf)^2\mu^1}(x+\mathrm{i}y)}.$$

Since the measures $(Uf)^2\mu^1_{\mathrm{sing}}$ and μ_{sing} are mutually singular, by Theorem 1 in Lecture 2,

$$\lim_{y\downarrow 0}\frac{P_{(Uf)^2\mu^1}(x+\mathrm{i}y)}{P_{\mu}(x+\mathrm{i}y)}=0 \qquad \text{for}\ \ \mu_{\mathrm{sing}}-a.e.\ x.$$

Hence,

$$\lim_{y\downarrow 0}\frac{\mathrm{Im}F_{(Uf)\mu^1}(x+\mathrm{i}y)}{\sqrt{P_{\mu^1}(x+\mathrm{i}y)}\sqrt{P_{\mu}(x+\mathrm{i}y)}}=0 \qquad \text{for}\ \ \mu_{\mathrm{sing}}-a.e.\ x. \tag{22}$$

Since

$$P_{\mu^1}(x+\mathrm{i}y)P_{\mu}(x+\mathrm{i}y)=\mathrm{Im}F_1(x+\mathrm{i}y)\mathrm{Im}F_0(x+\mathrm{i}y)=\frac{(\mathrm{Im}F_0(x+\mathrm{i}y))^2}{|1+F_0(x+\mathrm{i}y)|^2}\leq 1 \tag{23}$$

for all $x\in\mathbb{R}$, (22) yield (20).

To prove (21), note that

$$\begin{aligned}|L(x+\mathrm{i}y)|&=\frac{\mathrm{Im}F_{(Uf)\mu^1}(x+\mathrm{i}y)}{\sqrt{P_{\mu^1}(x+\mathrm{i}y)}\sqrt{P_{\mu}(x+\mathrm{i}y)}}\frac{|\mathrm{Re}(1+F_0(x+\mathrm{i}y))|}{\mathrm{Im}F_0(x+\mathrm{i}y)}\frac{\mathrm{Im}F_0(x+\mathrm{i}y)}{|1+F_0(x+\mathrm{i}y)|}\\&\leq\frac{\mathrm{Im}F_{(Uf)\mu^1}(x+\mathrm{i}y)}{\sqrt{P_{\mu^1}(x+\mathrm{i}y)}\sqrt{P_{\mu}(x+\mathrm{i}y)}}.\end{aligned}$$

Hence, (22) yields (21).

Going back to (19), applying Theorem 1 in Lecture 2 to the left-hand side and using relations (20) and (21) on the right-hand side we get that

$$\lim_{y\downarrow 0} F_{(Uf)\mu^1}(x+\mathrm{i}y) = f(x) \qquad \text{for } \mu_{\mathrm{sing}} - a.e.\ x. \tag{24}$$

Going back to the relation (18) we derive

$$\frac{F_{f\mu}(x+\mathrm{i}y)}{F_\mu(x+\mathrm{i}y)} = \left(\frac{1}{F_0(x+\mathrm{i}y)}+1\right) F_{(Uf)\mu^1}(x+\mathrm{i}y).$$

Since $|F_0(x+\mathrm{i}y)| \to \infty$ as $y \downarrow 0$ for μ_{sing}-a.e. x, the Poltoratskii theorem (in the special case $\nu_{\mathrm{s}} = 0$, μ compactly supported, $f \in L^2(\mathbb{R}, \mathrm{d}\mu)$ real valued) follows from (24).

We now remove the assumptions $f \in L^2(\mathbb{R}, \mathrm{d}\mu)$ and that f is real valued (we still assume that μ is compactly supported and that $\nu_{\mathrm{s}} = 0$). If $f \in L^1(\mathbb{R}, \mathrm{d}\mu)$ is *positive*, set $g = 1/(1+f)$, and $\mu = g\rho$. Then

$$\lim_{y\downarrow 0} \frac{F_\mu(x+\mathrm{i}y)}{F_{(1+f)\mu}(x+\mathrm{i}y)} = \lim_{y\downarrow 0} \frac{F_{g\rho}(x+\mathrm{i}y)}{F_\rho(x+\mathrm{i}y)} = \frac{1}{1+f(x)},$$

for μ_{sing}-a.e. x. By the linearity of the Borel transform,

$$\lim_{y\downarrow 0} \frac{F_{f\mu}(x+\mathrm{i}y)}{F_\mu(x+\mathrm{i}y)} = \lim_{y\downarrow 0} \frac{F_{(1+f)\mu}(x+\mathrm{i}y)}{F_\mu(x+\mathrm{i}y)} - 1 = f(x),$$

for μ_{sing}-a.e. x. Since every complex valued $f \in L^1(\mathbb{R}, \mathrm{d}\mu)$ is a linear combination of four positive functions in $L^1(\mathbb{R}, \mathrm{d}\mu)$, the linearity of Borel transform implies the statement for all $f \in L^1(\mathbb{R}, \mathrm{d}\mu)$.

Assume that μ is not compactly supported (we still assume $\nu_{\mathrm{s}} = 0$) and let $[a,b]$ be a finite interval. Decompose $\mu = \mu_1 + \mu_2$, where $\mu_1 = \mu \restriction [a,b]$, $\mu_2 = \restriction \mathbb{R} \setminus [a,b]$. Since

$$\frac{F_{f\mu}(z)}{F_\mu(z)} = \frac{F_{f\mu_1}(z) + F_{f\mu_2}(z)}{F_{\mu_1}(z)(1 + F_{\mu_2}(z)/F_{\mu_1}(z))}$$

and $\lim_{y\downarrow 0} |F_{\mu_1}(x+\mathrm{i}y)| \to \infty$ for $\mu_{1,\mathrm{sing}}$ a.e. $x \in [a,b]$,

$$\lim_{y\downarrow 0} \frac{F_{f\mu}(x+\mathrm{i}y)}{F_\mu(x+\mathrm{i}y)} = f(x) \qquad \text{for } \mu_{\mathrm{sing}}\text{-a.e. } x \in (a,b).$$

Since $[a,b]$ is arbitrary, we have removed the assumption that μ is compactly supported.

Finally, to finish the proof we need to show that if $\nu \perp \mu$, then

$$\lim_{y\downarrow 0} \frac{F_\nu(x+\mathrm{i}y)}{F_\mu(x+\mathrm{i}y)} = 0 \tag{25}$$

for $\mu_{\rm sing}$-a.e. x. Since ν can be written as a linear combination of four positive measures each of which is singular w.r.t. μ, w.l.o.g. me may assume that ν is positive. Let S be a Borel set such that $\mu(S)=0$ and that ν is concentrated on S. Then

$$\lim_{y\downarrow 0}\frac{F_{\chi_S(\mu+\nu)}(x+\mathrm{i}y)}{F_{\mu+\nu}(x+\mathrm{i}y)}=\chi_S(x),$$

for $\mu_{\rm sing}+\nu_{\rm sing}$-a.e. x. Hence,

$$\lim_{y\downarrow 0}\frac{F_{\nu}(x+\mathrm{i}y)}{F_{\mu}(x+\mathrm{i}y)+F_{\nu}(x+\mathrm{i}y)}=0$$

for $\mu_{\rm sing}$ a.e. x, and this yields (25). □

The Poltoratskii theorem of course also holds for complex measures μ:

Theorem 10.2. Let ν and μ be complex Borel measures and $\nu=f\mu+\nu_{\rm s}$ be the Radon-Nikodym decomposition. Let $|\mu|_{\rm sing}$ be the part of $|\mu|$ singular with respect to the Lebesgue measure. Then

$$\lim_{y\downarrow 0}\frac{F_{\nu}(x+\mathrm{i}y)}{F_{\mu}(x+\mathrm{i}y)}=f(x)\qquad\text{for }\ |\mu|_{\rm sing}-\ a.e.\,x.$$

Theorem 10.2 follows easily from Theorem 10.1. The details are left for homework.

Finally, we mention the following simple consequences of the Poltoratskii theorem:

Corollary 10.3. Let ν and μ be positive Borel measures and $\nu=f\mu+\nu_{\rm s}$ be the Radon-Nikodym decomposition. Let $\nu_{\rm s,sing}$ be the part of $\nu_{\rm s}$ singular with respect to the Lebesgue measure. Then

$$\lim_{y\downarrow 0}\frac{F_{\mu}(x+\mathrm{i}y)}{F_{\nu}(x+\mathrm{i}y)}=0\qquad\text{for }\ \nu_{\rm s,sing}-\ a.e.\,x.$$

Corollary 10.4. Let ν and μ be positive Borel measures. If

$$\liminf_{y\downarrow 0}\frac{|F_{\mu}(x+\mathrm{i}y)|}{|F_{\nu}(x+\mathrm{i}y)|}\neq 0\qquad\text{for }\ \nu_{\rm sing}-\ a.e.\,x,$$

then $\nu_{\rm sing}$ is absolutely continuous w.r.t. $\mu_{\rm sing}$.

The proofs of Corollaries 10.3 and 10.4 are simple and are left for homework.

11. Jaksic-Last theorems I

In this subsection we consider again the standard rank one perturbation theory framework described in the introduction. However, in this and the next subsection *we do not assume that ψ is a cyclic vector for H_0.*

Let $0 \neq \varphi \in \mathcal{H}$, and let μ_φ^λ be the spectral measure for H_λ and φ. In this and the next subsection we denote the spectral measure for H_λ and ψ by μ_ψ^λ. We wish to study the relation between μ_ψ^λ and μ_φ^λ. Of course, in general these two spectral measures may not be related at all. For example, if $\varphi \perp \mathcal{H}_\psi$, then μ_φ^λ is independent of λ and is equal to the spectral measure for H_0 and φ. Hence, the *minimal* assumption under which one may say something is:

Assumption 11.1 $\varphi \not\perp \mathcal{H}_\psi$.

This assumption is equivalent to:

Assumption 11.2 There exists $z \in \mathbb{C} \setminus \mathbb{R}$ such that $(\varphi|(H_0 - z)^{-1}\psi) \neq 0$.

We have introduced the notion of the essential support of the absolutely continuous spectrum in the Lecture 3 (please read again this definition). In analogous way, one defines the essential support of any positive Borel measure ν on $\mathbb{R}$ absolutely continuous w.r.t. the Lebesgue measure. More precisely, if $\mathrm{d}\nu = f(x)\mathrm{d}x$, then the essential support of ν, denoted $\mathrm{ess.supp}\,\nu$ is the class of equivalence associated to $\{x : f(x) > 0\}$.

In this subsection, we prove:

Theorem 11.3. Suppose that Assumption 11.1 holds. Then for Lebesgue a.e. $\lambda \in \mathbb{R}$,

$$\mathrm{ess.supp}\,\mu_{\psi,\mathrm{ac}}^\lambda \subset \mathrm{ess.supp}\,\mu_{\varphi,\mathrm{ac}}^\lambda.$$

In other words, for Lebesgue a.e. $\lambda \in \mathbb{R}$, the measure $\mu_{\psi,\mathrm{ac}}^\lambda$ is absolutely continuous w.r.t. $\mu_{\varphi,\mathrm{ac}}^\lambda$.

This theorem has been proven in [JL2]. As we shall see in Lectures 8 and 9, Theorem 11.3 has some striking consequences for the spectral theory of Anderson type Hamiltonians.

Proof. For $\phi_1, \phi_2 \in \mathcal{H}$ and $z \in \mathbb{C} \setminus \mathbb{R}$ we write $G_\lambda(\phi_1, \phi_2, z) = (\phi_1|(H_\lambda - z)^{-1}\phi_2)$. The identities (1) yield that

$$G_\lambda(\varphi, \psi, z) = \frac{G_0(\varphi, \psi, z)}{1 + \lambda G_0(\psi, \psi, z)}, \tag{26}$$

and

$$G_\lambda(\varphi, \varphi, z) = G_0(\varphi, \varphi, z) - \lambda G_\lambda(\varphi, \psi, z) G_0(\psi, \varphi, z). \tag{27}$$

Substituting, we get

$$G_\lambda(\varphi,\varphi,z) = G_0(\varphi,\varphi,z) - \lambda\frac{G_0(\varphi,\psi,z)G_0(\psi,\varphi,z)}{1+\lambda G_0(\psi,\psi,z)}. \tag{28}$$

Rearranging this identity we derive

$$|1+\lambda G_0(\psi,\psi,z)|^2 G_\lambda(\varphi,\varphi,z) = \lambda^2 a(z) + \lambda b(z) + c(z), \tag{29}$$

where

$$\begin{aligned} a(z) &= |G_0(\psi,\psi,z)|^2 G_0(\varphi,\varphi,z) - \overline{G}_0(\psi,\psi,z)G_0(\varphi,\psi,z)G_0(\psi,\varphi,z) \\ b(z) &= 2G_0(\varphi,\varphi,z)\mathrm{Re}G_0(\psi,\psi,z) - G_0(\varphi,\psi,z)G_0(\psi,\varphi,z) \\ c(z) &= G_0(\varphi,\varphi,z). \end{aligned} \tag{30}$$

Let

$$S := \{x\in\mathbb{R} \,:\, G_0(\phi_1,\phi_2,x),\ \phi_i = \psi \text{ or } \varphi,\ \text{exists and is finite and non-zero}\}. \tag{31}$$

Here, as usual, $G_0(\phi_1,\phi_2,x) = \lim_{y\downarrow 0} G_0(\phi_1,\phi_2,x+\mathrm{i}y)$. By Assumption 11.2 and Theorem 10 in Lecture 2, the set $\mathbb{R}\setminus S$ has zero Lebesgue measure. If $x\in S$, setting $z = x+\mathrm{i}y$ in (29) and taking $y\downarrow 0$ we derive

$$|1+\lambda G_0(\psi,\psi,x)|^2 \mathrm{Im}G_\lambda(\varphi,\varphi,x) = \lambda^2\mathrm{Im}a(x) + \lambda\mathrm{Im}b(x) + \mathrm{Im}c(x). \tag{32}$$

Let $\tilde S := S\cap\{x : \mathrm{Im}G_0(\psi,\psi,x) > 0\}$. Assume that $x\in\tilde S$ and that

$$\mathrm{Im}G_\lambda(\varphi,\varphi,x) = 0, \qquad \mathrm{Im}a(x) = 0, \qquad \mathrm{Im}b(x) = 0. \tag{33}$$

It then follows from (32) and the third relation in (30) that $\mathrm{Im}G_0(\varphi,\varphi,x) = 0$. The second relation in (30) yields

$$\mathrm{Im}(G_0(\varphi,\psi,x)G_0(\psi,\varphi,x)) = 0. \tag{34}$$

This relation and the first relation in (30) yield

$$\mathrm{Im}G_0(\psi,\psi,x)\mathrm{Re}(G_0(\varphi,\psi,x)G_0(\psi,\varphi,x)) = 0. \tag{35}$$

Since for $x\in S$, $G_0(\varphi,\psi,x)G_0(\psi,\varphi,x)\neq 0$, (34) implies that $\mathrm{Re}(G_0(\varphi,\psi,x)G_0(\psi,\varphi,x))\neq 0$. Thus, (35) implies $\mathrm{Im}G_0(\psi,\psi,x) = 0$, and this contradicts the choice of x. Therefore, if $\mathrm{Im}G_\lambda(\varphi,\varphi,x) = 0$ for some $x\in\tilde S$, we must have that either $a(x)\neq 0$ or $b(x)\neq 0$.

Let

$$\begin{aligned} T &:= \{(\lambda,x)\in\mathbb{R}\times\tilde S \,:\, \mathrm{Im}G_\lambda(\varphi,\varphi,x) = 0\}, \\ P(x) &:= \{\lambda\in\mathbb{R} \,:\, \mathrm{Im}G_\lambda(\varphi,\varphi,x) = 0\}, \\ Q(\lambda) &:= \{x\in\tilde S \,:\, \mathrm{Im}G_\lambda(\varphi,\varphi,x) = 0\}. \end{aligned}$$

Since $\mathrm{Im}G_\lambda(\varphi,\varphi,x)\geq 0$ for fixed $x\in\tilde{S}$, the function

$$\lambda\to\lambda^2\mathrm{Im}a(x)+\lambda\mathrm{Im}b(x)+\mathrm{Im}c(x),$$

is either identically equal to zero for all λ or has at most one zero. If either $\mathrm{Im}a(x)\neq 0$ or $\mathrm{Im}b(x)\neq 0$, then this function cannot be identically zero. Therefore, for $x\in\tilde{S}$, the set $P(x)$ consists of at most one point. By Fubini's theorem, the set T has Lebesgue measure zero, and then by Fubini's theorem again, for Lebesgue a.e. λ the set $Q(\lambda)$ has Lebesgue measure zero. Since $\mathrm{Im}G_\lambda(\varphi,\varphi,x)>0$ for $x\in\tilde{S}\setminus Q(\lambda)$, we derive that for Lebesque a.e. λ, $\mathrm{Im}G_\lambda(\varphi,\varphi,x)>0$ for Lebesgue a.e. $x\in\tilde{S}$. By Theorem 1 in Lecture 2,

$$\tilde{S}\subset\mathrm{ess.supp}\,\mu_\varphi^\lambda \tag{36}$$

for Lebesgue a.e. λ. On the other hand, by Theorem 1 in this lecture notes,

$$\mathrm{ess.supp}\,\mu_\psi^\lambda=\tilde{S} \tag{37}$$

for *all* λ. The statement follows from (36) and (37). □

12. Jaksic-Last theorems II

We continue with the same framework as in Subsection 11. In this subsection we will study the relation between $\mu_{\psi,\mathrm{sing}}^\lambda$ and $\mu_{\varphi,\mathrm{sing}}^\lambda$.

Theorem 12.1. Suppose that Assumption 11.1 holds. Then for Lebesgue a.e. λ the measure $\mu_{\psi,\mathrm{sing}}^\lambda$ is absolutely continuous w.r.t. $\mu_{\varphi,\mathrm{sing}}^\lambda$.

Combining Theorems 11.3 and 12.1 we derive

Corollary 12.2. Suppose that Assumption 11.1 holds. Then for Lebesgue a.e. λ the measure μ_ψ^λ is absolutely continuous w.r.t. μ_φ^λ.

Proof of Theorem 12.1. It follows from the relation (28) that

$$\frac{G_\lambda(\varphi,\varphi,z)}{G_\lambda(\psi,\psi,z)}=\frac{G_0(\varphi,\varphi,z)}{G_\lambda(\psi,\psi,z)}-\lambda\frac{G_0(\varphi,\psi,z)G_0(\psi,\varphi,z)}{G_0(\psi,\psi,z)}. \tag{38}$$

By Theorem 1 in Lecture 2,

$$\lim_{y\downarrow 0}|G_\lambda(\psi,\psi,x+\mathrm{i}y)|=\infty \tag{39}$$

for a.e. x w.r.t. $\mu_{\psi,\mathrm{sing}}^\lambda$. Let S be given by (31) and let $x\in S$. It follows from (38) and (39) that

$$\frac{G_\lambda(\varphi,\varphi,x)}{G_\lambda(\psi,\psi,x)}=-\frac{\lambda G_0(\varphi,\psi,x)G_0(\psi,\varphi,x)}{G_0(\psi,\psi,x)} \tag{40}$$

for a.e. x w.r.t. $\mu^\lambda_{\psi,\mathrm{sing}} \restriction S$. Since for $\lambda \neq 0$ the r.h.s. of (40) is different from zero, by Corollary 10.4 the measure $\mu^\lambda_{\psi,\mathrm{sing}} \restriction S$ is absolutely continuous w.r.t. $\mu^\lambda_{\varphi,\mathrm{sing}}$. By the spectral averaging

$$\int_{\mathbb{R}} \mu^\lambda_{\psi,\mathrm{sing}}(\mathbb{R} \setminus S)\mathrm{d}\lambda \leq \int_{\mathbb{R}} \mu^\lambda_{\psi}(\mathbb{R} \setminus S)\mathrm{d}\lambda = |\mathbb{R} \setminus S| = 0, \tag{41}$$

and so the measure $\mu^\lambda_{\psi,\mathrm{sing}}$ is concentrated on S for Lebesgue a.e. λ. Hence, for Lebesgue a.e. λ, the measure $\mu^\lambda_{\psi,\mathrm{sing}}$ is absolutely continuous w.r.t. $\mu^\lambda_{\varphi,\mathrm{sing}}$. $\square$

Theorem 12.1 was proven in [JL2]. Note that we have not used the full strength of the Poltoratskii theorem in its proof— we only needed Corollary 10.4. The strong form of the Poltoratskii theorem allows for an important improvement of Theorem 12.1. Recall that $\mathcal{H}_\psi$ denotes the cyclic subspace spanned by H_λ and ψ.

Theorem 12.2. Suppose that Assumption 11.1 holds and that for Lebesgue a.e. λ, the measures μ^λ_ψ and μ^λ_φ are equivalent. Then, for Lebesgue a.e. λ, $\mathbf{1}_{\mathrm{sing}}(H_\lambda)\varphi \in \mathcal{H}_\psi$.

Proof. By Theorem 1, for $\lambda \neq 0$ the measure $\mu^\lambda_{\psi,\mathrm{sing}}$ is concentrated on the set

$$S_1 := \{x \,:\, G_0(\psi, \psi, x) = -\lambda^{-1}\}.$$

The set S_1 does not need to be contained in S. However, by (41), for Lebesgue a.e. $\lambda \in \mathbb{R}$, $\mu^\lambda_{\psi,\mathrm{sing}}$ is concentrated on $S_1 \cap S$. Hence, for Lebesgue a.e. λ and for a.e x w.r.t. $\mu^\lambda_{\psi,\mathrm{sing}}$,

$$\begin{aligned}\lim_{y\downarrow 0} \frac{G_\lambda(\psi, \varphi, x+\mathrm{i}y)}{G_\lambda(\psi, \psi, x+\mathrm{i}y)} &= \lim_{y\downarrow 0} \frac{G_0(\psi, \varphi, x+\mathrm{i}y)}{G_0(\psi, \psi, x+\mathrm{i}y)} \\ &= \frac{G_0(\psi, \varphi, x)}{G_0(\psi, \psi, x)} = -\lambda G_0(\psi, \varphi, x).\end{aligned}$$

In the sequel ϕ_ψ denotes the projection of a vector ϕ on the cyclic subspace $\mathcal{H}_\psi$. We identify $\mathcal{H}_\psi = L^2(\mathbb{R}, \mathrm{d}\mu_\psi)$, $\psi = 1$. Then, by the Poltoratskii theorem 10.1,

$$\lim_{y\downarrow 0} \frac{G_\lambda(\psi, \varphi, x+\mathrm{i}y)}{G_\lambda(\psi, \psi, x+\mathrm{i}y)} = (\mathbf{1}_{\mathrm{sing}}(H_\lambda)\varphi)_\psi(x)$$

for a.e. x w.r.t. $\mu^\lambda_{\psi,\mathrm{sing}}$. This yields that for a.e. x w.r.t. $\mu^\lambda_{\psi,\mathrm{sing}}$ and for Lebesgue a.e. λ,

$$\left(\mathbf{1}_{\mathrm{sing}}(H_\lambda)\varphi\right)_\psi(x) = -\lambda G_0(\psi, \varphi, x). \tag{42}$$

Let

$$\mathrm{d}\mu^\lambda_\varphi = f_{\lambda,\varphi,\psi}\mathrm{d}\mu^\lambda_\psi,$$

be the Radon-Nikodym decomposition of μ_φ^λ w.r.t. μ_ψ^λ. It follows from the Poltoratskii theorem that for a.e. x w.r.t $\mu_{\psi,\mathrm{sing}}^\lambda$,

$$\lim_{y\downarrow 0}\frac{G_\lambda(\varphi,\varphi,x+\mathrm{i}y)}{G_\lambda(\psi,\psi,x+\mathrm{i}y)}=f_{\lambda,\varphi,\psi}(x).$$

The relation (40) yields that for a.e. x w.r.t. $\mu_{\psi,\mathrm{sing}}^\lambda$,

$$\begin{aligned}\frac{G_\lambda(\varphi,\varphi,x)}{G_\lambda(\psi,\psi,x)}&=-\lambda\frac{G_0(\varphi,\psi,x)G_0(\psi,\varphi,x)}{G_0(\psi,\psi,x)}\\&=\lambda^2G_0(\varphi,\psi,x)G_0(\psi,\varphi,x).\end{aligned}$$

It follows that for Lebesgue a.e. λ and for a.e. x w.r.t. $\mu_{\psi,\mathrm{sing}}^\lambda$,

$$f_{\lambda,\varphi,\psi}(x)=\lambda^2G_0(\varphi,\psi,x)G_0(\psi,\varphi,x).$$

Since $G_0(\psi,\psi,x)$ is real for a.e. x w.r.t. $\mu_{\psi,\mathrm{sing}}^\lambda$ (in fact, equal to $-\lambda^{-1}$),

$$G_0(\varphi,\psi,x)=\overline{G_0}(\psi,\varphi,x)\qquad \mu_{\psi,\mathrm{sing}}^\lambda-a.s.,$$

and

$$\begin{aligned}\mathrm{d}\mu_{\varphi,\mathrm{sing}}^\lambda(x)&=\lambda^2|G_0(\varphi,\psi,x)|^2\mathrm{d}\mu_{\psi,\mathrm{sing}}^\lambda(x)\\&=|(\mathbf{1}_{\mathrm{sing}}(H_\lambda)\varphi)_\psi(x)|^2\mathrm{d}\mu_{\psi,\mathrm{sing}}^\lambda(x),\end{aligned}$$

for Lebesgue a.e. λ (we have used (42)). Hence, for Lebesgue a.e. λ, $\mathbf{1}_{\mathrm{sing}}(H_\lambda)\varphi\in\mathcal{H}_\psi$. $\Box$

Theorem 12.2 has been proven in [JL3]. We will use this theorem in Lecture 9 to prove that the singular spectrum of Anderson type Hamiltonians is simple with probability one.

14. The Aizenman-Graf estimates

This subsection is a continuation of Subsection 8 and is based on [AG]. It should have followed right after Subsection 8 but I forgot to include it. We use the same framework and notation as in Subsection 8 and start with:

Lemma 14.1. Let A be a selfadjoint operator on a Hilbert space $\mathcal{H}$ and $\phi_1,\phi_2\in\mathcal{H}$ two unit vectors. Then for $z\in\mathbb{C}\setminus\mathbb{R}$ and $0\le s\le 2$,

$$|\mathrm{Im}z||(\phi_1|(A-z)^{-1}\phi_2)|^2\le|\mathrm{Im}(\phi_2|(A-z)^{-1}\phi_2)|\frac{|(\phi_1|(A-z)^{-1}\phi_2)|^s}{|(\phi_2|(A-z)^{-2}\phi_2)|^s}.\qquad(43)$$

Proof. For $s=0$ the estimate (43) follows from

$$|\mathrm{Im} z||(\phi_1|(A-z)^{-1}\phi_2)|^2 \le |\mathrm{Im} z|\|(A-z)^{-1}\phi_2\|^2 = |\mathrm{Im}(\phi_2|(A-z)^{-1}\phi_2)|. \tag{44}$$

For $s=2$, the estimate (43) follows from (44) with $\phi_1=\phi_2$. The general case follows by "interpolation". In this case "interpolation" means the following: the estimates for $s=0$ and $s=2$ can be written as

$$\frac{|\mathrm{Im} z||(\phi_1|(A-z)^{-1}\phi_2)|^2}{|\mathrm{Im}(\phi_2|(A-z)^{-1}\phi_2)|} \le 1, \tag{45}$$

$$\frac{|\mathrm{Im} z||(\phi_1|(A-z)^{-1}\phi_2)|^2}{|\mathrm{Im}(\phi_2|(A-z)^{-1}\phi_2)|} \le \frac{|\mathrm{Im}(\phi_1|(A-z)^{-1}\phi_2)|^2}{|(\phi_2|(A-z)^{-2}\phi_2)|^2} \tag{46}$$

Taking (45) to the power $1-s/2$, (46) to the power $s/2$, and multiplying the resulting estimates, we derive (43). □

Recall that P is a Borel probability measure on $\mathbb{R}$ with density p. $\mathbb{E}$ denotes the expectation w.r.t. P.

Theorem 14.2. Assume that $p \in L^\infty(\mathbb{R})$, $z \in \mathbb{C}\setminus\mathbb{R}$, and $0 \le s \le 2$. Then

$$|\mathrm{Im} z|\mathbb{E}\left(|(\varphi|(H_\lambda-z)^{-1}\psi)|^2\right) \le \pi\|p\|_\infty \mathbb{E}\left(\frac{|(\varphi|(H_\lambda-z)^{-1}\psi)|^s}{|(\psi|(H_\lambda-z)^{-1}\psi)|^s}\right) \tag{47}$$

Remark. The theorem is stated in the above form for the latter convenience. Note however that for all λ,

$$\frac{|(\varphi|(H_\lambda-z)^{-1}\psi)|^s}{|(\psi|(H_\lambda-z)^{-1}\psi)|^s} = \frac{|(\varphi|(H_0-z)^{-1}\psi)|^s}{|(\psi|(H_0-z)^{-1}\psi)|^s},$$

and so (47) is equivalent to

$$|\mathrm{Im} z|\mathbb{E}\left(|(\varphi|(H_\lambda-z)^{-1}\psi)|^2\right) \le \pi\|p\|_\infty \frac{|(\varphi|(H_0-z)^{-1}\psi)|^s}{|(\psi|(H_0-z)^{-1}\psi)|^s}.$$

Proof. We adopt the shorthand $G_\lambda(\phi_1,\phi_2,z) = (\phi_1|(H_\lambda-z)^{-1}\phi_2)$. By Lemma 14.1, for all λ

$$\begin{aligned}|\mathrm{Im} z||G_\lambda(\varphi,\psi,z)|^2 &\le |\mathrm{Im} G_\lambda(\psi,\psi,z)|\frac{|G_\lambda(\varphi,\psi,z)|^s}{|G_\lambda(\psi,\psi,z)|^s} \\ &= \frac{\mathrm{Im} G_0(\psi,\psi,z)}{|1+\lambda G_0(\psi,\psi,z)|^2}\frac{|G_0(\varphi,\psi,z)|^s}{|G_0(\psi,\psi,z)|^s}.\end{aligned}$$

(We have used the identity $G_\lambda(\,\cdot\,,\psi,z) = G_0(\,\cdot\,,\psi,z)(1+\lambda G_0(\psi,\psi,z))^{-1}$.) By (7),

$$\begin{aligned}|\mathrm{Im}z|\mathbb{E}(|G_\lambda(\varphi,\psi,z)|^2) &\le \pi\|p\|_\infty \frac{|G_0(\varphi,\psi,z)|^s}{|G_0(\psi,\psi,z)|^s}\\ &= \pi\|p\|_\infty \mathbb{E}\left(\frac{|G_\lambda(\varphi,\psi,z)|^s}{|G_\lambda(\psi,\psi,z)|^s}\right).\end{aligned}$$

□

We now extend Theorem 14.2 to more general densities.

Theorem 14.3. Assume that $p \in L^q(\mathbb{R})$ for some $q \in (1,\infty]$ and that $0 \le s \le 2$. Then:
(1) For all $z \in \mathbb{C}\setminus\mathbb{R}$,

$$|\mathrm{Im}z|^{1+1/q}\ \mathbb{E}\left(|(\varphi|(H_\lambda - z)^{-1}\psi)|^2\right) \le C\left(\mathbb{E}\ \frac{|(\varphi|(H_\lambda - z)^{-1}\psi)|^s}{|(\psi|(H_\lambda - z)^{-1}\psi)|^s}\right)^{(q-1)/q} \tag{48}$$

where $C = \pi^{1-1/q}\|p\|_q((q-1)^{-1/q} + (q-1)^{(q-1)/q})$.
(2) Let $[a,b]$ be a finite interval. Then for $y \neq 0$,

$$\int_a^b |y|\,\mathbb{E}(|(\varphi|(H_\lambda - x - \mathrm{i}y)^{-1}\psi)|^2)\mathrm{d}x \le D\left(\int_a^b \mathbb{E}\left(\frac{|(\varphi|(H_\lambda - x - \mathrm{i}y)^{-1}\psi)|^s}{|(\varphi|(H_\lambda - x - \mathrm{i}y)^{-1}\psi)|^s}\right)\mathrm{d}x\right)^{(q-1)/q} \tag{49}$$

where $D = \pi(b-a)^{1-1/q}\|p\|_q((q-1)^{-1/q} + (q-1)^{(q-1)/q})$.

Remark. The remark after Theorem 14.2 also applies to Theorem 14.3.

Proof. (1) Set

$$c = \frac{|(\varphi|(H_0 - z)^{-1}\psi)|^s}{|(\psi|(H_0 - z)^{-1}\psi)|^s} = \mathbb{E}\left(\frac{|(\varphi|(H_\lambda - z)^{-1}\psi)|^s}{|(\psi|(H_\lambda - z)^{-1}\psi)|^s}\right).$$

For $\delta > 0$ write $\mathbb{E}_{\underline{\delta}}(X) = \int_{\{\lambda:p(\lambda)\le\delta\}} X\mathrm{d}P$, $\mathbb{E}_{\overline{\delta}}(X) = \mathbb{E}(X) - \mathbb{E}_{\underline{\delta}}(X)$. By Theorem 14.2,

$$|\mathrm{Im}z|\mathbb{E}_{\underline{\delta}}\left(|(\varphi|(H_\lambda - z)^{-1}\psi)|^2\right) \le c\pi\delta. \tag{50}$$

We estimate

$$\begin{aligned}|\mathrm{Im}z|\mathbb{E}_{\overline{\delta}}\left(|(\varphi|(H_\lambda - z)^{-1}\psi)|^2\right) &\le |\mathrm{Im}z|^{-1}P(\{\lambda : p(\lambda) > \delta\})\\ &\le |\mathrm{Im}z|^{-1}\delta^{q-1}\|p\|_q^q.\end{aligned} \tag{51}$$

Combining (50) and (51) and optimizing over δ we derive (1).
(2) Set

$$d = \int_a^b \frac{|G_0(\varphi, \psi, x+\mathrm{i}y)|^s}{|G_0(\psi, \psi, x+\mathrm{i}y)|^s}\mathrm{d}x = \int_a^b \mathbb{E}\left(\frac{|G_\lambda(\varphi, \psi, x+\mathrm{i}y)|^s}{|G_\lambda(\psi, \psi, x+\mathrm{i}y)|^s}\right)\mathrm{d}x.$$

By (50),

$$\int_a^b |y|\mathbb{E}_{\underline{\delta}}\left(|(\varphi|(H_\lambda - x - \mathrm{i}y)^{-1}\psi)|^2\right)\mathrm{d}x \leq d\pi\delta(b-a). \tag{52}$$

On the other hand, if P_{μ_ψ} is the Poisson transform of the spectral measure μ_ψ,

$$\int_a^b |y||(\varphi|(H_\lambda - x - \mathrm{i}y)^{-1}\psi)|^2\mathrm{d}x \leq \int_{\mathbb{R}} P_{\mu_\psi}(x+\mathrm{i}|y|)\mathrm{d}x = \pi,$$

and so

$$\begin{aligned}\int_a^b |y|\mathbb{E}_{\underline{\delta}}\left(|(\varphi|(H_\lambda - x - \mathrm{i}y)^{-1}\psi)|^2\right)\mathrm{d}x &\leq \pi P(\{\lambda : p(\lambda) > \delta\}) \\ &\leq \pi\delta^{1-q}\|p\|_q^q.\end{aligned} \tag{53}$$

Combining (52) and (53) and optimizing over δ we derive the statement. □

15. Homework

[1] Prove relation (6).
Hint: See Theorem I.2 in [Si1].

[2] Prove Theorem 5.3.

[3] Check whether I got the right constant C in Theorem 8.2.

[4] Prove Theorem 8.3.

[5] Prove Theorem 10.2.

[6] Prove Corollaries 10.3 and 10.4.

[7] Prove the Poltoratskii theorem 10.1 in the case where ν and μ are positive pure point measures.

[8] In the Poltoratskii theorem one cannot replace μ_{sing} with μ. Find an example justifying this claim. Is there any nontrivial condition under which μ_{sing} can be replaced with μ?

[9] Prove that Assumption 11.1 is equivalent to Assumption 11.2. Prove that Assumption 11.1 holds iff for some $z \in \mathbb{C}_+$, $(\varphi|(H_0 - z)^{-1}\psi) \neq 0$.

[10] (1) For each $0 < \tau < 1$ construct an example of τ-regular singular continuous Borel probability measure on $\mathbb{R}$. What happens for $\tau = 1$?

(2) Find Borel probability measure on $\mathbb{R}$ with density which is not τ-regular for any $0 < \tau \leq 1$.

14. Project

This project deals with various examples involving rank one Anderson type Hamiltonians. Note that these Hamiltonians are completely determined by a choice of a Borel probability measure μ^0 on $\mathbb{R}$. Setting $\mathcal{H} = L^2(\mathbb{R}, \mathrm{d}\mu^0)$, H_0 = operator of multiplication by x, $\psi \equiv 1$, we obtain a class of rank one Anderson type Hamiltonians $H_\lambda = H_0 + \lambda(\psi|\cdot)\psi$. By the spectral theorem, up to unitary equivalence, any rank one Anderson type Hamiltonian restricted to the cyclic subspace $\mathcal{H}_\psi$ has this form.

[1] (1) Please read Example 3 in Section I.4. of Reed-Simon I. This example discuses the construction of the Cantor measure μ_C . The Cantor measure is the canonical example of a singular continuous measure.
(2) Let $\mathrm{d}\mu^0 = (\mathrm{d}x \restriction [0,1] + \mathrm{d}\mu_C)/2$. The ac spectrum of H_0 is $[0,1]$. The singular continuous part of μ^0 is concentrated on the Cantor set C. Since C is closed, $\mathrm{sp}_{\mathrm{sing}}(H_0) = C$. Prove that for $\lambda \neq 0$ the spectrum of H_λ in $[0,1]$ is purely absolutely continuous.
Hint: See the last example in Section XIII.7 of Reed-Simon IV.

[2] Assume that $\mu^0 = \mu_C$. Prove that for all $\lambda \neq 0$, H_λ has only pure point spectrum. Compute the spectrum of H_λ.
Hint: This is Example 1 in [SW]. See also Example 3 in Section II.5 of [Si1].

[3] Let

$$\mu_n = 2^{-n} \sum_{j=1}^{2^n} \delta(j/2^n),$$

and $\mu = \sum_n a_n \mu_n$, where $a_n > 0$, $\sum_n a_n = 1$. The spectrum of H_0 is pure point and equal to $[0,1]$. Prove that the spectrum of H_λ in $[0,1]$ is purely singular continuous for all $\lambda \neq 0$. What is the spectrum of H_λ outside $[0,1]$?
Hint: This is Example 2 in [SW]. See also Example 4 in Section II.5 of [Si1].

[4] Let $\nu_{j,n}(A) = \mu_C(A + j/2^n)$.

$$\mu^0 = c\chi_{[0,1]} \sum_{n=1}^{\infty} n^{-2} \sum_{j=1}^{2^n} \nu_{j,n},$$

where c is the normalization constant. Prove that $\mathrm{sp}(H_0) = [0,1]$. Prove that this spectrum is purely singular continuous. Prove that the spectrum of H_λ in $[0,1]$ is purely singular continuous for all λ.
Hint: This is Example 5 in Section II.5 of [Si1].

[5] Find μ^0 such that the spectrum of H_0 is purely absolutely continuous and equal to $[0,1]$ and:
(1) For a set of λ's of positive Lebesgue measure, H_λ has embedded point spectrum in $[0,1]$.
(2) For a set of λ's of positive Lebesgue measure, H_λ has embedded singular continuous spectrum in $[0,1]$.
Hint: See Examples 7 and 8 in Section II.5 of [Si1].

[6] Del Rio and Simon [DS] have shown that there exists μ^0 such that:
(a) $\mathrm{sp}_{\mathrm{ac}}(H_\lambda) = [0,1]$.
(b) For a set of λ's of positive Lebesgue measure, H_λ has embedded point spectrum in $[0,1]$.
(c) For a set of λ's of positive Lebesgue measure, H_λ has embedded singular continuous spectrum in $[0,1]$.

[7] del Rio-Fuentes-Poltoratskii [DFS] have shown that there exists μ^0 such that:
(a) $\mathrm{sp}_{\mathrm{ac}}(H_\lambda) = [0,1]$. Moreover, for all $\lambda \in [0,1]$, the spectrum of H_λ is purely absolutely continuous.
(b) For all $\lambda \notin [0,1]$, $[0,1] \subset \mathrm{sp}_{\mathrm{sing}}(H_\lambda)$.

[8] Let μ^0 be a pure point measure with atoms $\mu^0(x_n) = a_n$, $n \in \mathbb{N}$. Clearly, $\sum_n a_n = 1$ and

$$G(x) = \sum_{n=1}^{\infty} \frac{a_n}{(x - x_n)^2}.$$

(1) Prove that if $\sum_n \sqrt{a_n} < \infty$, then $G(x) < \infty$ for Lebesgue a.e. $x \in [0,1]$.
(2) Assume that $x_n = x_n(\omega)$ are independent random variables uniformly distributed on $[0,1]$ (we keep a_n deterministic). Assume that $\sum_n \sqrt{a}_n = \infty$. Prove that for a.e. ω, $G(x) = \infty$ for Lebesgue a.e. $x \in [0,1]$.
(3) What can you say about the spectrum of H_λ in the cases (1) and (2)?
Hint: (1) and (2) are proven in J. Howland: Perturbation theory of dense point spectra, J. Func. Anal. 74, 52-80, 1987.