MATH 743
SPECTRAL THEORY OF ANDERSON TYPE HAMILTONIANS
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LECTURE 6

Anderson type Hamiltonians

O.1 References

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1.1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space and H_0 a bounded selfadjoint operator on $\mathcal{H}$. Let $\{\delta_n\}_{n\in\Gamma}$ be a set of orthonormal vectors in $\mathcal{H}$. Let $\Omega = \mathbb{R}^\Gamma$, and for each $\omega \in \Omega$, let

$$H_\omega = H_0 + \sum_{n\in\Gamma} \omega(n)(\delta_n|\cdot)\delta_n. \tag{1}$$

We will often use the shorthand

$$V_\omega := \sum_{n\in\Gamma} \omega(n)(\delta_n|\cdot)\delta_n,$$

and write $H_\omega = H_0 + V_\omega$. Since H_0 is bounded, H_ω is selfadjoint for all ω and $\mathrm{Dom}(H_\omega) = \mathrm{Dom}(V_\omega)$.

The family of selfadjoint operators $\{H_\omega\}_{\omega\in\Omega}$ we will call *the Anderson type Hamiltonians* [JL1]. They are a conceptually and technically convenient generalization of many specific models of discrete random Schrödinger operators discussed in the literature.

Let $\mathcal{F}$ be the Borel σ-algebra in Ω. It is not difficult to show that the map $\omega \to H_\omega$ is selfadjoint measurable. The model is specified by the choice of a probability measure P on $(\Omega, \mathcal{F})$.

I will usually try to make optimal assumptions on the measure P. In the beginning, you should always keep in mind the simple case where P is a product measure and $\{\omega(n)\}_{n\in\Gamma}$ are independent random variables.

The assumption that H_0 is bounded is not necessary. In your first project you are asked to extend the results of this lecture to unbounded H_0 (of course, you will need to formulate some conditions on H_0—for example, you need to ensure that H_ω is selfadjoint for P-a.e. ω etc.)

In this lecture we will discuss some basic properties of Anderson type Hamiltonians. The Subsections 1.2, 1.3 and 1.4 are partly based on the papers [JL1, JL2]. In Subsection 1.4 we prove the decomposition theorem of [JL2]. This theorem will play an important role in our study of the spectral structure of Anderson type Hamiltonians. The proof of the decomposition theorem is somewhat complex and introduces technical devices which will be used on many occasions in the course. In Subsection 1.5 we introduce ergodic Anderson type Hamiltonians and prove celebrated theorems of Pastur. In Subsections 1.6 and 1.7 we briefly discuss Anderson type Hamiltonians on graphs. In Subsection 1.7 we introduce the Anderson model which is the best known and most studied example of the Anderson type Hamiltonians. The proofs in this subsection are based on Section 4 in [JL3].

The proofs in this lecture are compact and follow the style of the research literature. It is very important that you understand all the missing details.

There are three projects at the end of notes. The first deals with various technical extensions of the results in the notes. The second deals with the concept of the discrete Laplacian on $\mathbb{Z}^d$ and the third deals with deterministic discrete Schrödinger operators (these two projects are still in the draft form). A large amount of important material is condensed in these two projects.

1.2. Cyclicity and measurability

Let $\mathcal{H}_{\omega,\Gamma}$ be the cyclic space generated by H_ω and $\{\delta_n\}_{n\in\Gamma}$. Recall that $\mathcal{H}_{\omega,\Gamma}$ is the closure of the linear span of the set of vectors

$$\{(H_\omega - z)^{-1}\delta_n \,:\, z \in \mathbb{C}\setminus\mathbb{R}, n \in \Gamma\}.$$

Theorem 2.1. $\mathcal{H}_{\omega,\Gamma}$ is equal to the cyclic subspace spanned by H_0 and $\{\delta_n\}_{n\in\Gamma}$ and hence independent of ω.

Proof. Let $\mathcal{H}_\Gamma$ be the cyclic subspace generated by H_0 and $\{\delta_n\}_{n\in\Gamma}$. Since $(H_\omega - z)^{-1}\delta_n \in \mathrm{Dom}(V_\omega)$ for all $\omega \in \Omega$ and $z \in \mathbb{C}\setminus\mathbb{R}$, we have

$$(H_\omega - z)^{-1}\delta_n = (H_0 - z)^{-1}\delta_n - \sum_{p\in\Gamma}\omega(p)(\delta_p|(H_\omega - z)^{-1}\delta_n)(H_0 - z)^{-1}\delta_p, \tag{2}$$

and so $\mathcal{H}_{\omega,\Gamma} \subset \mathcal{H}_\Gamma$. On the other hand, for $k = 0, 1, \cdots$,

$$\lim_{y\to\infty} y^2 \left((H_\omega + \mathrm{i}y)^{-1} - (V_\omega + \mathrm{i}y)^{-1}\right) H_0^k\delta_n = H_0^{k+1}\delta_n. \tag{3}$$

By induction, for all $k \geq 0$, $H_0^k\delta_n \in \mathcal{H}_{\omega,\Gamma}$. Hence, $\mathcal{H}_\Gamma \subset \mathcal{H}_{\omega,\Gamma}$. □

From this theorem it follows that in the study of Anderson type Hamiltonians w.l.o.g. we may replace $\mathcal{H}$ with the cyclic subspace spanned by H_0 and $\{\delta_n\}_{n\in\Gamma}$.

We denote by μ_n^ω the spectral measure for H_ω and δ_n, and by $\mathcal{H}_{\omega,n}$ the cyclic subspace spanned by H_ω and δ_n. Let $a_n > 0$, $\sum_n a_n < \infty$. Then

$$\mu^\omega = \sum_{n\in\Gamma} a_n\mu_n^\omega \tag{4}$$

is a spectral measure for H_ω. In the sequel, we will often refer to (4) as *the* spectral measure for H_ω.

Theorem 2.2. The function $\omega \to H_\omega$ is selfadjoint measurable.

Proof. For all $\varphi, \psi \in \mathcal{H}$ and all n the function

$$\omega \to (\varphi| \left(\mathrm{e}^{\mathrm{i}H_0t/n}\mathrm{e}^{\mathrm{i}V_\omega t/n}\right)^n \psi)$$

is measurable. By the Trotter product formula

$$(\varphi|\mathrm{e}^{\mathrm{i}tH_\omega}\psi) = \lim_{n\to\infty}(\varphi| \left(\mathrm{e}^{\mathrm{i}H_0t/n}\mathrm{e}^{\mathrm{i}V_\omega t/n}\right)^n \psi),$$

and so, for all t, the function $\omega \to \exp(\mathrm{i}tH_\omega)$ is measurable. By the homework problem 1 in Lecture 5, $\omega \to H_\omega$ is selfadjoint measurable. □

1.3. Some consequences of Kolmogorov's 0-1 law.

By enumarating Γ, in this subsection w.l.o.g. we may assume that $\Gamma = \{1, \cdots, N\}$ or $\Gamma = \mathbb{N}$.

Definition 3.1. A probability measure P is called *trivial at infinity* if either Γ is finite or the tail σ-field of the sequence $\{\omega(n)\}_{n\in\mathbb{N}}$ is trivial (note that this condition does not depend on the choice of enumaration of Γ; if it holds for one enumaration, then it holds for all enumerations).

$\Sigma_{\rm ac}^{{\rm ess},\omega}$ denotes the essential support of the a.c. spectrum of H_ω (the corresponding class of equivalence).

Theorem 3.2. Assume that P is trivial at infinity. Then there exist deterministic Borel sets $\Sigma_{\rm ac}$ and $\Sigma_{\rm ac}^{\rm ess}$ such that:
(1) ${\rm sp}_{\rm ac}(H_\omega)=\Sigma_{\rm ac}$ P-a.s.
(2) $\Sigma_{\rm ac}^{{\rm ess},\omega}=\Sigma_{\rm ac}^{\rm ess}$ P-a.s.

Proof. We need to consider only the case $\Gamma=\mathbb{N}$ (the theorem is trivial if Γ is finite).
(1) Let μ^ω be the spectral measure for H_ω (given by (4)). For $r_1,r_2\in\mathbb{Q}$, $r_1<r_2$, set

$$E_{r_1,r_2}=\{\omega\,:\,(r_1,r_2)\cap{\rm supp}\,\mu_{\rm ac}^\omega=\emptyset\}.$$

In other words $\omega\in E_{r_1,r_2}\Leftrightarrow\mu_{\rm ac}^\omega\{(r_1,r_2)\}=0\Leftrightarrow(r_1,r_2)\cap{\rm sp}_{\rm ac}(H_\omega)=\emptyset$. The set E_{r_1,r_2} is measurable. Since the absolutely continuous spectrum is invariant under finite rank perturbations (this follows from the Kato-Rosenblum theorem in Lecture 3—for a simple proof see Lecture 7), if $\omega\in E_{r_1,r_2}$, then for all n,

$$\{\omega'\in\Omega:\omega'(k)=\omega(k)\ {\rm for}\,k\geq n\}\subset E_{r_1,r_2}$$

Hence, for all n, E_{r_1,r_2} is measurable w.r.t. $\sigma(\{\omega(k)\}_{k\geq n})$, and so E_{r_1,r_2} belongs to the tail σ-field τ. Hence, $P(E_{r_1,r_2})\in\{0,1\}$. Let

$$\Omega_{r_1,r_2}=\begin{cases}E_{r_1,r_2} & \text{if } P(E_{r_1,r_2})=1,\\ \Omega\setminus E_{r_1,r_2} & \text{if } P(E_{r_1,r_2})=0,\end{cases}$$

and

$$\tilde\Omega=\bigcap_{r_1,r_2}\Omega_{r_1,r_2}.$$

Note that $P(\tilde\Omega)=1$. We claim that for $\omega_1,\omega_2\in\tilde\Omega$, ${\rm sp}_{\rm ac}(H_{\omega_1})={\rm sp}_{\rm ac}(H_{\omega_2})$. Indeed, if $e\not\in{\rm sp}_{\rm ac}(H_{\omega_1})$, then there are rational numbers $r_1<e<r_2$ such that $\omega_1\in E_{r_1,r_2}$. It follows that $\omega_2\in E_{r_1,r_2}$ and $e\not\in{\rm sp}_{\rm ac}(H_{\omega_2})$. By reversing roles of ω_1 and ω_2 we derive the statement.
(2) For $r\in\mathbb{Q}$, let

$$F_\omega(r)=\int_{-\infty}^r(1+t^2)^{-1}\chi_{\Sigma_{\rm ac}^{{\rm ess},\omega}}(t){\rm d}t.$$

The function $\omega\to F_\omega(r)$ is measurable (Lecture 5). Since the essential support of the absolutely continuous spectrum of H_ω is invariant under finite rank perturbations (this follows from the Kato-Rosenblum theorem in Lecture 3—for a simple proof see Lecture 7), the function $\omega \to F_\omega(r)$ is measurable w.r.t. the tail σ-field τ. Hence, for all $r \in \mathbb{Q}$ there exists a constant $\alpha(r)$ such that $F_\omega(r) = \alpha(r)$ for P-a.e. ω. Since $x \to F_\omega(x)$ is continuous, there exists a *deterministic* function $x \to \alpha(x)$ such that $F_\omega(x) = \alpha(x)$ for all $x \in \mathbb{R}$ and P-a.e. ω. The function $\alpha(x)$ is differentiable for Lebesgue a.e. x (since $F_\omega(x)$ is) and let $f(x) = \alpha'(x)$. Then f is deterministic and

$$f(x) = (1 + x^2)^{-1} \chi_{\Sigma_{\rm ac}^{{\rm ess},\omega}}(x)$$

as elements of $L^1(\mathbb{R}, \mathrm{d}x)$. In particular, the deterministic set $\Sigma_{\rm ac}^{\rm ess} = \{x : f(x) > 0\}$ is an essential support of the ac spectrum of H_ω for P-a.e. ω. □

Theorem 3.3. Assume that P is trivial at infinity. Then there exist Borel set $\Sigma_{\rm ess}$ such that ${\rm sp}_{\rm ess}(H_\omega) = \Sigma_{\rm ess}$ P-a.s.

Proof. Again, you need to consider only the case $\Gamma = \mathbb{N}$. For $r_1, r_2 \in \mathbb{Q}$, $r_1 < r_2$, set

$$E_{r_1,r_2} = \{\omega \,:\, \dim \mathbf{1}_{(r_1,r_2)}(H_\omega) < \infty\}.$$

In other words $\omega \in E_{r_1,r_2}$ iff $(r_1, r_2) \cap {\rm sp}_{\rm ess}(H_\omega) = \emptyset$. Prove that E_{r_1,r_2} is a measurable set and that $P(E_{r_1,r_2}) \in \{0, 1\}$. Finish the proof following the argument we used in the proof of Theorem 3.2 (1). The details are left for the homework. □

1.4. The decomposition theorem

Definition 4.1. We say that the measure P is *conditionaly absolutely continuous (a.c)* if for all n the conditional probability distribution of $\omega(n)$, given $\{\omega(m)\}_{m \neq n}$, is absolutely continuous.

Recall that $\mathcal{H}_{\omega,n}$ is the cyclic subspace spanned by H_ω and δ_n.

Theorem 4.2. Assume that P is trivial at infinity and conditionally a.c. Then:
(1) For $n, m \in \Gamma$ set $E_{n,m} := \{\omega : \mathcal{H}_{\omega,n} \not\perp \mathcal{H}_{\omega,m}\}$. Then $P(E_{n,m}) \in \{0, 1\}$.
(2) The relation $n \sim m$ iff $P(E_{n,m}) = 1$ is a relation of equivalence on Γ.

Proof. (1) Note first that $\mathcal{H}_{\omega,n} \not\perp \mathcal{H}_{\omega,m}$ iff there exists $z \in \mathbb{C} \setminus \mathbb{R}$ such that $(\delta_n|(H_\omega - z)^{-1}\delta_m) \neq 0$. Let $\omega \in E_{n,m}$ and let $z \in \mathbb{C} \setminus \mathbb{R}$ be such that $(\delta_n|(H_\omega - z)^{-1}\delta_m) \neq 0$. Let $p \in \Gamma$ and

$$H_{\omega,\lambda} = H_\omega + \lambda(\delta_p|\,\cdot\,)\delta_p, \qquad \lambda \in \mathbb{R}.$$

Then

$$\begin{aligned}(\delta_n|(H_{\omega,\lambda}-z)^{-1}\delta_m) &= (\delta_n|(H_\omega-z)^{-1}\delta_m)-\lambda(\delta_n|(H_\omega-z)^{-1}\delta_p)(\delta_p|(H_{\omega,\lambda}-z)^{-1}\delta_m)\\ &= (\delta_n|(H_\omega-z)^{-1}\delta_m)-\lambda\frac{(\delta_n|(H_\omega-z)^{-1}\delta_p)(\delta_p|(H_\omega-z)^{-1}\delta_m)}{1+\lambda(\delta_p|(H_\omega-z)^{-1}\delta_p)}.\end{aligned} \tag{5}$$

We have used in this derivation simple rank one perturbation formulas. You can either prove them (they are elementary) or consult Lecture 7. It follows from (5) that if $\omega\in E_{n,m}$ and z is such that $(\delta_n|(H_\omega-z)^{-1}\delta_m)\neq 0$, then there exists at most one $\lambda\in\mathbb{R}$ such that $(\delta_n|(H_{\omega,\lambda}-z)^{-1}\delta_m)=0$. Let now $p_1,\cdots,p_k$ be distinct, and

$$H_{\omega,\lambda_1,\cdots,\lambda_k}=H_\omega+\sum_{j=1}^{k}\lambda_j(\delta_{p_j}|\,\cdot\,)\delta_{p_j}.$$

Arguing inductively, show that if $\omega\in E_{n,m}$ and $(\delta_n|(H_\omega-z)^{-1}\delta_m)\neq 0$, then for Lebesgue a.e. $(\lambda_1,\cdots,\lambda_k)\in\mathbb{R}^k$, $(\delta_n|H_{\omega,\lambda_1,\cdots,\lambda_k}-z)^{-1}\delta_m)\neq 0$. If Γ is finite, this completes the proof. If $\Gamma=\mathbb{N}$, we need to apply the conditional Fubini theorem. Fix k and let $P_{\underline{k}}$ be the conditional probability distribution of $\{\omega(j)\}_{j\leq k}$ given $\{\omega(j)\}_{j>k}$. Let $P_{\overline{k}}$ be the restriction of P to $\sigma(\{\omega(j)\}_{j>k})$. Let $\Omega_1=\mathbb{R}^k$, $\Omega_2=\mathbb{R}^{\Gamma\setminus[1,k]}$. We denote the variables in $\Omega=\Omega_1\times\Omega_2$ by $(\omega_{\underline{k}},\omega_{\overline{k}})$.

By the conditional Fubini theorem, for any Borel $B\subset\mathbb{R}^k$

$$P(E_{n,m}\cap(\omega_{\underline{k}}\in B))=\int_{\Omega_2}\left(\int_B\chi_{E_{n,m}}(\omega_{\underline{k}},\omega_{\overline{k}})\mathrm{d}P_{\underline{k}}^{\omega_{\overline{k}}}(\omega_{\underline{k}})\right)\mathrm{d}P_{\overline{k}}(\omega_{\overline{k}}). \tag{6}$$

Now, if $\chi_{E_{n,m}}(\omega_{\underline{k}},\omega_{\overline{k}})=1$ for some $\omega=(\omega_{\underline{k}},\omega_{\overline{k}})$, then for Lebesgue a.e. $\omega_{\underline{k}}\in\mathbb{R}^k$, $\chi_{E_{n,m}}(\omega_{\underline{k}},\omega_{\overline{k}})=1$. Since for $P_{\overline{k}}$-a.e. $\omega_{\overline{k}}$ the probability measure $P_{\underline{k}}^{\omega_{\overline{k}}}$ is absolutely continuous w.r.t. the Lebesgue measure in $\mathbb{R}^k$, we have that for $P_{\overline{k}}$ -a.e. $\omega_{\overline{k}}$,

$$\int_B\chi_{E_{n,m}}(\omega_{\underline{k}},\omega_{\overline{k}})\mathrm{d}P_{\underline{k}}^{\omega_{\overline{k}}}(\omega_{\underline{k}})=P_{\underline{k}}^{\omega_{\overline{k}}}(B)\int_{\Omega_1}\chi_{E_{n,m}}(\omega_{\underline{k}},\omega_{\overline{k}})\mathrm{d}P_{\underline{k}}^{\omega_{\overline{k}}}(\omega_{\underline{k}}). \tag{7}$$

Assume now that B has the form

$$B=I_{r_1}\times\cdots I_{r_l}\times\mathbb{R}^{\Gamma\setminus[1,l]}. \tag{8}$$

where r_j's are rationals and $l<k$ is fixed. Then,

$$P_k^{\omega}(B):=P^{\omega_{\overline{k}}}(B)=\mathbb{E}(\chi_B|\sigma(\{\omega(j)\}_{j>k}))(\underline{\omega}),$$

where we used the usual identification discussed in Subsection 1.9 of Lecture 4. Hence, by the martingale convergence theorem (for the negatively indexed martingales, Theorem 15-6 H in Burrill), the limit

$$\lim_{k\to\infty}P_k^{\omega}(B)=:Y(\omega), \tag{9}$$

exists P-a.s. and Y is measurable w.r.t. the tail σ-field τ. Hence, $Y = \text{const}\ P - a.s.$, and since

$$\int_\Omega P_k^\omega(B)\mathrm{d}P = \int_{\Omega_2} P^{\omega_{\overline{k}}}(B)\mathrm{d}P_{\overline{k}}(\omega_{\overline{k}}) = P(B),$$

$Y = P(B)$ P-a.s. Combining (6) and (7) with the dominated convergence theorem we derive that for any B of the form (8),

$$P(E_{nm}\cap B) = \lim_{k\to\infty} P(E_{n,m}\cap(\omega_{\underline{k}}\in B)) = P(E_{nm})P(B).$$

The set of B's such that $P(E_{n,m}\cap P(B) = P(E_{n,m})P(B)$ is a σ-algebra. Since the sets of the form (8) generate $\mathcal{F}$, $P(E_{n,m}\cap P(B) = P(E_{n,m})P(B)$ for all $B\in\mathcal{F}$. Taking $B = E_{nm}$ we derive $P(E_{nm}) = P(E_{nm})^2$, and so $P(E_{n,m})\in\{0,1\}$.
(2) Assume that $n\sim p$ and $p\sim m \Leftrightarrow P(E_{n,p}\cap E_{p,m}) = 1$. Let $\omega\in E_{n,p}\cap E_{p,m}$. Let z be such that $(\delta_n|(H_\omega - z)^{-1}\delta_p)(\delta_p|(H_\omega - z)^{-1}\delta_m)\neq 0$. Then (5) again yields that there exists at most one λ such that $(\delta_n|(H_{\omega,\lambda} - z)^{-1}\delta_m) = 0$. Let $\delta^{(p)}$ be the Kronecker delta, which is an element of the probability space $\Omega = \mathbb{R}^\Gamma$. Then for every $\omega\in E_{n,p}\cap E_{p,m}$, $\omega + \lambda\delta^{(p)}\in E_{n,m}$ for Lebesgue a.e. λ.

To simplify the notation in the sequel we assume that $p = 1$. We will use the decomposition of the probability space Ω described in Subsection 1.9 of Lecture 4. We will also follow the notation of that subsection except that we denote the conditional probability distribution of $\omega(1)$ given $\{\omega(j)\}_{j\geq 2}$ by P^{ω_2}. By the conditional Fubini theorem

$$P((E_{n,p}\cap E_{p,m})\setminus E_{n,m}) = \int_{\Omega_2}\left[\int_{\Omega_1}\chi_{((E_{n,p}\cap E_{p,m})\setminus E_{n,m})}(\omega_1,\omega_2)\mathrm{d}P^{\omega_2}(\omega_1)\right]\mathrm{d}P_2(\omega_2).$$

Since the measure P^{ω_2} is absolutely continuous w.r.t. the Lebesgue measure for P_2-a.e. ω_2, $P((E_{n,p}\cap E_{p,m})\setminus E_{n,m}) = 0$. Hence, $P(E_{n,m}) = 1$ and $n\sim m$. □

We finish this section with some remarks concerning Theorem 4.2. Consider the following example: $\mathcal{H} = L^2([0,1],\mathrm{d}x)$, $\Gamma = \{1,2,3\}$, $H_0 =$ operator of multiplication by x, $\delta_1 = \sqrt{2}\chi_{[0,1/2]}$, $\delta_3 = \sqrt{2}\chi_{[1/2,1]}$, $\delta_2 = \chi_{[0,1/4]} - \chi_{[1/4,3/4]} + \chi_{[3/4,1]}$. Let H_ω, $\omega = (\omega_1,\omega_2,\omega_3)$, be the corresponding Anderson type Hamiltonians. The first observation is that if $\omega_2 = 0$, then for all $z\in\mathbb{C}\setminus\mathbb{R}$ and all ω_1,ω_2,

$$(\delta_1|(H_\omega - z)^{-1}\delta_3) = 0.$$

If P has density, then Theorem 4.2 holds. However, in this example Theorem 4.2 is not a deterministic statement, namely it is not true that for all ω, $\mathcal{H}_{\omega,1}\not\perp\mathcal{H}_{\omega,3}$. Note also that if $P = \nu_1\otimes\nu_2\otimes\nu_3$, and $\mathrm{d}\nu_1 = \mathrm{d}\nu_3 = \mathrm{d}x\restriction[0,1]$, and $\nu_2(\{0\}) = \nu_2(\{1\}) = 1/2$, then $P(E_{1,3}) = 1/2$, and so Theorem 4.2 may fail if the random variables do not have densities. In your first project you are asked to study whether the condition that P is trivial at infinity can be relaxed.

In some examples, however, Theorem 4.2 indeed holds for all ω. A perhaps most important class of such examples are the Anderson type Hamiltonians on graphs (which include the standard Anderson model on $\mathbb{Z}^d$). This class of Hamiltonians is discussed in Subsections 1.6 and 1.7.

1.5. Some consequences of ergodicity

We start by description of ergodic Anderson type Hamiltonians (we will not consider the most general case—see the monographs [CL] and [PF] for additional information about random ergodic selfadjoint operators). We will assume that $\Gamma = \mathbb{Z}^d$ for some $d \geq 1$. Let $T^j, j \in \mathbb{Z}^d$, be the usual shift operator on $\Omega = \mathbb{R}^\Gamma$,

$$(T^j\omega)(n) = \omega(n - j).$$

The map T^j is a bijection and $E \in \mathcal{F}$ iff $T^j(E) \in \mathcal{F}$ for all j. Note also that the collection of maps $\{T^j\}_{j\in\mathbb{Z}^d}$ is a group.

Denote by $\ell^2(\Gamma)$ the subspace spanned by the orthonormal set $\{\delta_n\}_{n\in\Gamma}$. Let U_j, $j \in \mathbb{Z}^d$, be the unitary operators on $\mathcal{H}$ defined by $U_j\delta_n = \delta_{n-j}$ and $U_j\psi = \psi$ if $\psi \in \ell^2(\Gamma)^\perp$. Note that the collection of unitary maps $\{U_j\}_{j\in\mathbb{Z}^d}$ is a unitary group.

A probability measure P on $(\Omega, \mathcal{F})$ is called stationary if for all j and E, $P(T^j(E)) = P(E)$. A stationary probability measure is called *ergodic* iff $T^j(E) = E \bmod 0$ for all j, then $P(E) \in \{0, 1\}$.

Assume that P is ergodic and let X be an extended random variable such that $X \circ T^j = X$ P-a.s. for all $j \in \mathbb{Z}^d$. Then $X = \text{const}$ P-a.s. The proof of this result is the same as the proof Theorem 1.6.1 in Lecture 4 (see also the proof of Proposition 9.1 in [CFKS]).

Definition 5.1 The Anderson type Hamiltonians $H_\omega = H_0 + V_\omega$ are called *ergodic* if P is ergodic and for all j, $U_jH_0U_j^* = H_0$.

If $\{H_\omega\}_{\omega\in\Omega}$ are ergodic Anderson type Hamiltonians, then for all j and ω,

$$U_jH_\omega U_j^* = H_{T^j\omega}. \tag{10}$$

Until the end of this subsection we will consider ergodic Anderson type Hamiltonians. Theorems 5.2–5.4 below are due to Leonid Pastur and have played an important role in the development of the subject.

Theorem 5.2. There exist deterministic sets $\Sigma_{\text{ac}}, \Sigma_{\text{sc}}, \Sigma_{\text{pp}}$ such that for P-a.e. ω,

$$\text{sp}_{\text{ac}}(H_\omega) = \Sigma_{\text{ac}}, \qquad \text{sp}_{\text{sc}}(H_\omega) = \Sigma_{\text{sc}}, \qquad \text{sp}_{\text{pp}}(H_\omega) = \Sigma_{\text{pp}}.$$

In particular, if $\Sigma = \Sigma_{\text{ac}} \cup \Sigma_{\text{sc}} \cup \Sigma_{\text{pp}}$, then for P-a.e. ω, $\text{sp}(H_\omega) = \Sigma$.

Proof. Let μ^ω be the spectral measure for H_ω. Note that

$$\mu^{T^j\omega} = \sum_{n\in\mathbb{Z}^d} a_n \mu_n^{T^j\omega} = \sum_{n\in\mathbb{Z}^d} a_n \mu_{n+j}^{\omega}.$$

Hence, the measures $\mu^{T^j\omega}$ and μ^ω are equivalent for *all* $\omega \in \Omega$. In particular, their ac, sc, and pp components are also equivalent measures.

In what follows $\#$ stands for ac or pp or sc. Recall that $\mathrm{sp}_\#(H_\omega) = \mathrm{supp}\,\mu_\#^\omega$. For $r_1, r_2 \in \mathbb{Q}$, $r_1 < r_2$, consider the event

$$E_{r_1,r_2} = \{\omega \,:\, (r_1, r_2) \cap \mathrm{supp}\,\mu_\#^\omega = \emptyset\}.$$

Since $T^j(E_{r_1,r_2}) = E_{r_1,r_2}$ for all j and P is ergodic, $P(E_{r_1,r_2}) \in \{0,1\}$. The rest of the proof is the same as in Theorem 3.2 (1). □

Theorem 5.3. Let B a Borel set. Then $\dim 1_B(H_\omega) = 0$ P-a.s. or $\dim 1_B(H_\omega) = \infty$ P-a.s.

Proof. Consider the sum

$$S(\omega) := \sum_{n\in\mathbb{Z}^d} (\delta_n | \mathbf{1}_B(H_\omega)\delta_n).$$

If $k_\omega := \dim \mathbf{1}_B(H_\omega) < \infty$ and $\{\phi_{j,\omega}\}_{1\le j\le k_\omega}$ is an orthonormal basis of $\mathrm{Ran}\mathbf{1}_B(H_\omega)$, then

$$S(\omega) = \sum_{n\in\mathbb{Z}^d} \sum_{j=1}^{k_\omega} |(\delta_n|\phi_{j,\omega})|^2 \le k_\omega.$$

The random variable $S(\omega)$ is invariant under T^j for all j and hence P-a.s. constant. The monotone convergence theorem yields that for P-a.e. ω,

$$\begin{aligned} S(\omega) &= \lim_{N\to\infty} \sum_{|j|\le N} \mathbb{E}((\delta_0|\mathbf{1}_B(H_{T^j\omega})\delta_0)) \\ &= \lim_{N\to\infty} N\mathbb{E}\left((\delta_0|\mathbf{1}_B(H_\omega)\delta_0)\right). \end{aligned}$$

If $S(\omega) \ne \infty$ P-a.s., then $\mathbb{E}((\delta_0|\mathbf{1}_B(H_\omega)\delta_0)) = 0$ P-a.s., and so $S(\omega) = 0$ P-a.s. This implies that $\mathbf{1}_B(H_\omega)\delta_n = 0$ P-a.s., and since $\{\delta_n\}$ is a cyclic set for H_ω, $\mathbf{1}_B(H_\omega) = 0$ P-a.s. □

Theorem 5.4 $\mathrm{sp}_{\mathrm{disc}}(H_\omega) = \emptyset$ P-a.s.

Proof. Since $\mathrm{sp}(H_\omega) = \Sigma$ P-a.s., the discrete spectrum of H_ω is with probability one contained in the set of isolated points of Σ (this set is obviously countable). Since for any isolated point $e \in \Sigma$, $\dim \mathbf{1}_e(H_\omega) = \infty$ P-a.s., $\mathrm{sp}_{\mathrm{disc}}(H_\omega) = \emptyset$ P-a.s. □

Theorem 5.5. There exists a deterministic set $\Sigma_{\mathrm{ac}}^{\mathrm{ess}}$ such that $\Sigma_{\mathrm{ac}}^{\mathrm{ess},\omega} = \Sigma_{\mathrm{ac}}^{\mathrm{ess}}$ P-a.s.

Proof. The proof is very similar to the proof of Theorem 3.2 (2). For $r \in \mathbb{Q}$ set

$$F_\omega(r) = \int_{-\infty}^{r} (1+t^2)^{-1} \chi_{\Sigma_{\mathrm{ac}}^{\mathrm{ess},\omega}}(t)\mathrm{d}t.$$

Since the measures $\mu_{\mathrm{ac}}^{\omega}$ and $\mu_{\mathrm{ac}}^{T^j\omega}$ are equivalent, $\Sigma_{\mathrm{ac}}^{\mathrm{ess},\omega} = \Sigma_{\mathrm{ac}}^{\mathrm{ess},T^j\omega}$. Hence, for for all j and all $\omega \in \Omega$,

$$F_{T^j\omega}(r) = F_\omega(r).$$

It follows that every rational number r there exists deterministic constant $\alpha(r)$ such that $F_\omega(r) = \alpha(r)$ P-a.s. The rest of the proof is the same as in Theorem 3.2 (2). □

1.6. Anderson type Hamiltonians on graphs

Let $\mathcal{G}$ be the set of vertices of a simple connected countable graph and let $\rho : \mathcal{G} \times \mathcal{G} \to \{0, 1, \cdots\}$ be the distance on the graph. The function ρ is a metric. Note that ρ fully determines the edges of the graph (which can be identified with unordered pairs $(n, m) \in \mathcal{G} \times \mathcal{G}$ having $\rho(n, m) = 1$), and so we can think of the graph as being the metric space $(\mathcal{G}, \rho)$. We will assume that the degree of the points in the graph is bounded, namely that

$$\gamma := \sup_n \#\{m \in \mathcal{G} \,:\, \rho(m, n) = 1\} < \infty.$$

The discrete Laplacian on $\ell^2(\mathcal{G})$ is defined by

$$(\Delta_{\mathcal{G}}\psi)(n) = \sum_{\rho(m,n)=1} \psi(m). \tag{11}$$

$\Delta_{\mathcal{G}}$ is a bounded selfadjoint operator and $\|\Delta_{\mathcal{G}}\| \le \gamma$. If $\{\delta_n\}_{n\in\mathcal{G}}$ is the standard orthonormal basis of $\ell^2(\mathcal{G})$ ($\delta_n(m) = 1$ if $n = m$, 0 otherwise), then

$$\Delta_{\mathcal{G}} = \sum_{n\in\mathcal{G}} \sum_{\rho(m,n)=1} (\delta_m|\,\cdot\,)\delta_n.$$

Let $\Gamma \subset \mathcal{G}$ and let $\{H_\omega\}$ be the corresponding Anderson type Hamiltonians,

$$H_\omega = -\Delta_{\mathcal{G}} + \sum_{n\in\Gamma} \omega(n)(\delta_n|\,\cdot\,)\delta_n.$$

For obvious reasons, we will refer to H_ω as Anderson type Hamiltonians on the graph $\mathcal{G}$. This type of Anderson type Hamiltonians is most common in the literature.

For Anderson type Hamiltonians on graphs the Decomposition Theorem 4.2 reduces to a simple deterministic result.

Theorem 6.1 Let $n \neq m \in \mathcal{G}$ and let $l = \rho(n, m)$. Then for all $\omega \in \Omega$,

$$(\delta_m | H_\omega^l \delta_n) = (-1)^l l_{nm},$$

where l_{nm} is the number of path of length l connecting n and m. In particular, $\mathcal{H}_{\omega,n} \not\perp \mathcal{H}_{\omega,m}$ for all $\omega \in \Omega$.

Proof. First, note that $(\delta_m | \Delta_\mathcal{G}^l \delta_n) = l_{nm}$ (you can prove this by induction), and that $(\delta_m | \Delta_\mathcal{G}^k \delta_n) = 0$ for $k < l$. Similarly, if $p \in \mathcal{G}$ is such that $\rho(n, p) < l$, then $(\delta_m | H_0^k \delta_p) = 0$ if $k < l - \rho(n, p)$. Expanding $(-\Delta_\mathcal{G} + V_\omega)^l \delta_n$ we can write $(\delta_m | (H_0 + V_\omega)^l \delta_n)$ as a sum of the term $(-1)^l (\delta_m | \Delta_\mathcal{G}^l \delta_n) > 0$ and of the terms

$$(-1)^{\sum k_{2j+1}} (\delta_m | \Delta_\mathcal{G}^{k_1} V_\omega^{k_2} \cdots \Delta_\mathcal{G}^{k_{2j-1}} V_\omega^{k_{2j}} \delta_n), \tag{12}$$

where $k_i \geq 0$, $\sum k_i = l$, and at least one of the k_{2i}'s is not zero. Since each of the terms (12) is equal to zero, the statement follows. □

1.7. The Anderson model on $\mathbb{Z}^d$

Consider the graph $\mathcal{G} = \mathbb{Z}^d$ and let Δ be the corresponding discrete Laplacian on $\ell^2(\mathbb{Z}^d)$. The spectrum of Δ is purely absolutely continuous and equal to $[-2d, 2d]$. In your second project you are asked to prove these (and many others) properties of the free Laplacian on $\mathbb{Z}^d$.

Consider the Anderson type Hamiltonians on the graph $\mathbb{Z}^d$ with $\Gamma = \mathbb{Z}^d$. Assume that the probability measure P has the product form

$$P = \bigotimes_{n \in \mathbb{Z}^d} \nu, \tag{13}$$

where ν is a Borel probability measure on $\mathbb{R}$. Hence, the random variables $\{\omega(n)\}_{n \in \mathbb{Z}^d}$ are i.i.d. and the measure ν is their common probability distribution.

The family of Anderson type Hamiltonians $H_\omega = -\Delta + V_\omega$ together with the probability measure P is called *the Anderson model*. This is the best known and most studied example of the Anderson type Hamiltonians.

The Anderson model is an example of ergodic Anderson type Hamiltonians (you are asked in the homework to prove that the measure (13) is ergodic w.r.t. the shifts T^j). In particular, there exists deterministic set Σ such that $\mathrm{sp}(H_\omega) = \Sigma$ P-a.s. In this subsection we prove

Theorem 7.1 $\Sigma = [-2d, 2d] + \mathrm{supp}\, \nu$.

For example, if ν is Gaussian or Cauchy, then $\Sigma = \mathbb{R}$. If $\mathrm{d}\nu = (b-a)^{-1}\chi_{[a,b]}\mathrm{d}x$, then $\Sigma = [-2d + a, 2d + b]$.

Since $\mathrm{sp}(V_\omega) = \mathrm{supp}\,\nu$ P-a.s. (prove this), Theorem 7.1 asserts that $\mathrm{sp}(H_\omega) = \mathrm{sp}(-\Delta) + \mathrm{sp}(V_\omega)$ P-a.s. One direction of this identity follows from a simple deterministic statement:

Proposition 7.2. Let A and B be selfadjoint operators on a Hilbert space $\mathcal{H}$. Assume that A is bounded and that $\mathrm{sp}(A)$ is a connected set. Then $\mathrm{sp}(A+B) \subset \mathrm{sp}(A) + \mathrm{sp}(B)$.

Proof. By adding a constant to A we may assume that $\mathrm{sp}(A) = [-\|A\|, \|A\|]$. Let z be real and assume that $z \notin \mathrm{sp}(A) + \mathrm{sp}(B)$. Then $\mathrm{dist}(z, \mathrm{sp}(B)) > \|A\|$ and

$$\|(z-B)^{-1}A\| \leq \frac{\|A\|}{\|z-B\|} < 1.$$

The identity

$$z - (A+B) = (z-B)(1 - (z-B)^{-1}A),$$

yields that $z \notin \mathrm{sp}(A+B)$. $\square$

The proof of the other inclusion we split into a number of steps. For $n = (n_1, \cdots, n_d) \in \mathbb{Z}^d$ we set $|n| = \max |n_j|$.

Lemma 7.3. Let $a \in \mathrm{supp}\,\nu$ be given and let $d_k > 0$, $d_k \to 0$ be a given sequence. Then P-a.s. there exist a sequence of (ω-dependent) disjoint boxes

$$B_k = \{n \in \mathbb{Z}^d \,:\, |n - c_k| \leq L_k\}$$

where $|c_k| \to \infty$, $L_k \to \infty$, such that

$$\forall n \in B_k \qquad |\omega(n) - a| < d_k.$$

Proof. This lemma follows easily from the Borel 0-1 law and is left for homework. $\square$

Set $H_a = -\Delta + a$, $a \in \mathbb{R}$. Obviously, $\mathrm{sp}(H_a) = [-2d + a, 2d + a]$.

Proposition 7.4. Let $a \in \mathrm{supp}\,\nu$. Then for all $y \neq 0$ and $e \in \mathbb{R}$,

$$\|(H_\omega - e - \mathrm{i}y)^{-1}\| \geq \|(H_a - e - \mathrm{i}y)^{-1}\| \qquad P-a.s. \tag{14}$$

Proof. Let ψ be a unit vector in $\ell^2(\mathbb{Z}^d)$. Let ω be given and let B_k be the sequence of boxes from Lemma 7.3. Let c_k be the center of B_k,

$$\psi_k := U_{c_k}^* \psi, \qquad V_\omega^{(k)} := U_{c_k} V_\omega U_{c_k}^* = V_{T^{c_k}\omega},$$

and $H_\omega^{(k)} = -\Delta + V_\omega^{(k)}$. Since for every $n \in \mathbb{Z}^d$,

$$\lim_{k\to\infty} V_\omega^{(k)}(\delta_n) = a, \tag{15}$$

we have

$$\lim_{k\to\infty} (H_\omega^{(k)} - z)^{-1}\psi = (H_a - z)^{-1}\psi. \tag{16}$$

The proofs of (15) and (16) are left for homework. The identity

$$(H_\omega - e - \mathrm{i}y)^{-1}\psi_k = U_{-c_k}(H_\omega^{(k)} - e - \mathrm{i}y)^{-1}\psi$$

yields that

$$\lim_{k\to\infty} \|(H_\omega - e - \mathrm{i}y)^{-1}\psi_k\| = \|(H_a - e - \mathrm{i}y)^{-1}\psi\|.$$

Hence,

$$\|(H_\omega - e - \mathrm{i}y)^{-1}\| \geq \|(H_a - e - \mathrm{i}y)^{-1}\psi\|,$$

for all unit vectors ψ, and this yields the statement. $\square$

Proof of Theorem 7.1. It remains to show that $[-2d, 2d] + \operatorname{supp}\nu \subset \Sigma$. By Proposition 7.4, for all $a \in \operatorname{supp}\nu$, $[-2d, 2d] + a \subset \mathrm{sp}(H_\omega)$ P-a.s., and so $[-2d, 2d] + a \subset \Sigma$. $\square$

1.8. Homework

[1] Prove relations (2) and (3).

[2] Read the proof of the Trotter product formula in [RS1].

[3] Prove formula (5).

[4] Prove that the measure (13) is ergodic.

[5] Prove Lemma 7.3.

[6] Prove that (14) implies $[-2d, 2d] + a \subset \mathrm{sp}(H_\omega)$ P-a.s.
(Hint: Prove first that $e \in \mathrm{sp}(A) \Leftrightarrow \|(A - e - \mathrm{i}y)^{-1}\| = |y|^{-1}$ for all $y \neq 0$.)

[7] Prove relations (15) and (16).

[8] Prove Theorem 3.3.

1.9. Project I.

[1] In the definition of the Anderson type Hamiltonians we have assumed that H_0 is a bounded operator. By making appropriate assumptions extend the results of Subsections

1.1-1.5 to unbounded H_0. Notice that boundedness of H_0 has been explicitely used in the proof of Theorem 2.1—you will need to reprove this theorem.

[2] Study the proof of Theorem 4.2. Reprove this theorem in the case where the random variables $\{\omega(n)\}$ are independent and have densities (in this case, the proof is considerably simpler and you do not need to use martingale convergence theorems). Either exhibit an example where Theorem 4.2 fails if P is not trivial at infinity *or* prove the theorem under more general assumptions.

[3] Show that Proposition 7.2 holds if A is an arbitrary bounded selfadjoint operator and $\mathrm{sp}(B)$ is connected. If neither $\mathrm{sp}(A)$ nor $\mathrm{sp}(B)$ is connected, do you think this theorem is still true? Can you prove a variant of this theorem using, for example, the numerical range theorem?

[4] Please read pages 241-257 in [CL] and Chapter 1 in [PF]. Do all the exercises in Chapter 1 of [PF].

1.9. Project II: Discrete Laplacian on $\mathbb{Z}^d$

This project deals with the discrete Laplacian on $\mathbb{Z}^d$. We will denote it by Δ. Recall that

$$(\Delta\psi)(n) = \sum_{|m-n|=1} \psi(m),$$

where $|n| = \max |n_j|$.

Before doing this project, you should recall the properties of the continuous Laplace operator

$$\Delta_{\mathrm{c}} := \sum_{j=1}^{d} \partial_{x_j}^2$$

on $L^2(\mathbb{R}^d)$. In particular, please read again Section IX.7 in [RS2] and Lemma 1 in Appendix to XI.6 in [RS3].

In the first part of the project we will study properties of Δ. You should always compare them with the properties of Δ_{c}. In the second part of the project we will discuss some conceptual questions concerning the definition of the discrete Laplacian.

Let $\mathbb{T}^d = [0, 2\pi)^d$ be the usual d-dimensional torus. The variables on the torus we denote by $\theta = (\theta_1, \cdots, \theta_d)$. $\mathrm{d}\theta$ denotes the Lebesgue measure on $\mathbb{T}^d$. The Fourier transform

$$\hat{\psi}(\theta) = (2\pi)^{-d/2} \sum_{n\in\mathbb{Z}^d} \psi(n)\mathrm{e}^{-\mathrm{i}n\cdot\theta},$$

is a unitary map between $\ell^2(\mathbb{Z}^d)$ and $L^2(\mathbb{T}^d, \mathrm{d}\theta)$.

[1] Let $\Phi(\theta) := 2\sum_{j=1}^d \cos\theta_j$. Prove that

$$\widehat{(\Delta\psi)}(\theta) = \Phi(\theta)\hat{\psi}(\theta).$$

Conclude that the spectrum of Δ is purely absolutely continuous and equal to $[-2d, 2d]$. Describe the spectral measure for Δ and δ_n and the corresponding cyclic subspace the best you can.

[2] For $d = 3$, using Mathematica or Maple, plot the surfaces $\Phi(\theta) = 5$ and $\Phi(\theta) = 1$. Can you describe in mathematical terms the crucial difference between the pictures? For $d = 2$ plot the surfaces $\Phi(\theta) = 3$ and $\Phi(\theta) = 1$.

[3] Prove that for $0 < e < 2d$ the curvature of the surface $\Phi(\theta) = e$ is

$$K(\theta) = \frac{\sum_{j=1}^d \sin^2\theta_j \prod_{m=1, m\neq j}^d \cos\theta_j}{\left(\sum_{j=1}^d \sin^2\theta_j\right)^{\frac{d+1}{2}}}.$$

Can you use this formula to explain the pictures in [2]? Prove that the surface $\Phi(\theta) = e$ is strictly convex if $2d-4 < |e| < 2d$ (i.e. that $K(\theta) > 0$). Prove that the same surface is not convex if $0 < |e| < 2d-4$. (In the continuous case, the constant energy surface, the sphere $|k|^2 = e$, is convex for $e > 0$).

[4] Let $e \in [-2d, 2d]$ and let $\psi : \mathbb{Z}^d \to \mathbb{R}$ be a function such that $\Delta\psi = e\psi$. What can you say about ψ? Classify all polynomially bounded solutions. What happens if $e \notin [-2d, 2d]$? Compare with [13] and [14] in the project III.

[5] Let

$$R(n, z) := (\delta_0|(\Delta - z)^{-1}\delta_n) = (2\pi)^{-d}\int_{\mathbb{T}^d} \frac{\mathrm{e}^{\mathrm{i}n\cdot\theta}}{\Phi(\theta) - z}\mathrm{d}\theta, \qquad \mathrm{Im} z \neq 0. \tag{17}$$

Let $w = (w_1, \cdots, w_d)$ be the coordinates in $\mathbb{C}^d$ and consider the function

$$\Phi(w) = \sum_{j=1}^d w_j + w_j^{-1}.$$

This function is holomorhic in the region $\mathbb{C}^d \setminus \{w : w_j = 0 \text{ for some } j\}$. With a slight abuse of notation we denote $\mathbb{T}^d = \{w : |w_j| = 1 \text{ for all } j\}$. We also write $w^n = w_1^{n_1}\cdots w_d^{n_d}$, $1 = (1, \cdots, 1)$ etc. The parametrization $w_j = \exp(\mathrm{i}\theta_j)$ yields that

$$R(n, z) = (-\mathrm{i})^d \int_{\mathbb{T}^d} \frac{w^{n-1}}{\Phi(w) - z}\mathrm{d}w. \tag{18}$$

By interchanging the region of integration show that there exist constants $C_z > 0$ and $D_z > 0$ such that

$$|R(n, z)| < C_z \mathrm{e}^{-D_z|n|}. \tag{19}$$

What is the best estimate you can get on C_z and D_z (for an alternative proof of (19) see [8] in Project III).

[6] (Continuation of [5]). Assume that $|z| > 2d$. Let $\gamma = \ln(|z|/2d)$. Prove that

$$|R(n, z)| \leq \frac{1}{|z| - 2d} \mathrm{e}^{-\gamma|n|}.$$

Hint (i.e solution): Write

$$(\delta_0|(z - \Delta)^{-1}\delta_n) = z^{-1} \sum_{j=0}^{\infty} (\delta_0|(\Delta/z)^j \delta_n).$$

Since $(\delta_0|\Delta^j \delta_n) = 0$ for $j < |n|$,

$$|R(n, z)| \leq \sum_{j=|n|}^{\infty} |z|(2d/|z|)^j = \frac{1}{|z| - 2d} \mathrm{e}^{-\gamma|n|}.$$

[7] (Continuation of [5]). Compute $R(n, z)$ for $d = 1$.

In the next set of problems we will study discrete Laplacian with the help of Bessel functions. For $k \in \mathbb{Z}$ the Bessel function of order k, $J_k(x)$, is defined by

$$J_k(x) = \int_0^{2\pi} \mathrm{e}^{\mathrm{i}k\phi} \mathrm{e}^{-\mathrm{i}x\cos\phi} \mathrm{d}\phi.$$

All the results and estimates you will need can be found in G.N. Watson: *A Treatise on the Theory of Bessel Functions*, second edition, Cambrdige University Press (1995). In particular, the following estimates hold (we write $\langle y \rangle := (1 + y^2)^{1/2}$):

$$\sup_k |J_k(x)| \leq c_1 \langle x \rangle^{-1/3}, \qquad \sup_x |J_k(x)| \leq c_2 \langle k \rangle^{-1/3}. \tag{20}$$

An elegant proof of these estimates can be found in L. Landau: Bessel Functions: Monotonicity and Bounds, Journal of the London Mathematical Society **61** (2000) 197-215, http://www.mth.kcl.ac.uk/~llandau/Homepage/Homepage2.html
(Landau also obtains the the optimal values for the constants c_1 and c_2).

For our purposes, it is convenient to define $J_n(x)$ for $n = (n_1, \cdots, n_d) \in \mathbb{Z}^d$ by

$$J_n(x) = \prod_{j=1}^{d} J_{n_i}(x).$$

[8] The free propagator on the lattice is defined by

$$K_t(m, n) := (\delta_m | \mathrm{e}^{\mathrm{i}t\Delta} \delta_n) = J_{n-m}(-2t).$$

(For comparison, recall that the free propagator in the continuous case is given by

$$K_t(x, y) = (4\pi \mathrm{i}t)^{-d/2} \mathrm{e}^{\mathrm{i}|x-y|/4t},$$

were $|x|$ is the usual norm on $\mathbb{R}^d$.)

Set $K_t(n) := K_t(0, n)$. Some basic estimates on $K_t(n)$ can be derived from the asymptotics of the Bessel functions.
(1) $\sup_t |K_t(n)| \le c \prod \langle n_i \rangle^{-1/3}$ and this estimate is optimal.
(2) $\sup_n |K_t(n)| \le c \langle t \rangle^{-d/3}$ and this estimate is optimal.
(3) Compare this estimates with the continuous case where $|K_t(x)| = (4\pi|t|)^{-d/2}$.

[9] (Continuation of [8]) Using [8] we can derive some simple estimate on $R(n, z)$ (recall (17)).
(1) Prove that for $z \in \mathbb{C}_+$,

$$R(n, z) = \frac{1}{\mathrm{i}} \int_0^{\infty} \mathrm{e}^{\mathrm{i}tz} K_{-t}(n) \mathrm{d}t.$$

Using this formula and [8] show that:
(2) For $d \ge 5$ the function $\mathbb{C}_+ \ni z \to R(n, z)$ has a continuous extension to $\overline{\mathbb{C}}_+$ and satisfies

$$\sup_{z \in \overline{\mathbb{C}}_+} |R(n, z)| \le C \langle |n| \rangle^{-1/3}.$$

(3) For $d = 4$ the function $\mathbb{C}_+ \ni z \to R(n, z)$ has a continuous extension to $\overline{\mathbb{C}}_+$ and satisfies

$$\sup_{z \in \overline{\mathbb{C}}_+} |R(n, z)| \le C_\epsilon \langle |n| \rangle^{-1/3+\epsilon}.$$

for all $\epsilon > 0$.
(4) In (2) and (3), what can you say about the regularity of $e \to R(n, e)$? How does it depend on d?

(5) For $d = 1$, $R(n, z)$ does not decay for $z \in [-2, 2]$. We will study the decay in $d = 2, 3$ using different methods.

[10] Assume that $d = 2$.
(1) Assume that $e \neq 0, \pm 4$. Prove that the limit

$$R(n, e) := \lim_{y \downarrow 0} R(n, e + \mathrm{i}y)$$

exists. Prove that $\lim_{y \downarrow 0} \mathrm{Im} R(0, 4 + \mathrm{i}y) = \infty$.
(2) Let $2 > \delta > 0$ and $I_\delta = [-4 + \delta, -\delta] \cup [\delta, 4 - \delta]$. Prove that for some c independent of n,

$$\sup_{e \in I_\delta, y \geq 0} |R(n, e + \mathrm{i}y)| \leq c \langle n \rangle^{-1/2}.$$

Hint: A proof can be based on the formula (17). Consider the curve

$$\gamma(e) = \{\theta = (\theta_1, \theta_2) \in \mathbb{T}^2 \ : \ \Phi(\theta) = 2\cos\theta_1 + 2\cos\theta_2 = e\}.$$

(a) Prove that γ is a simple closed curve. Prove that

$$C_\delta := \inf_{\theta \in \gamma(e), e \in I_\delta} |\nabla \Phi(\theta)| > 0.$$

Conclude that for any interval $I \subset I_\delta$,

$$\big|\{\theta \in \mathbb{T}^2 \ : \ \Phi(\theta) \in I\}\big| \leq C_\delta |I|.$$

(Here $|A|$ denotes the Lebesgue measure of A).
(b) (Change of variable (based on the work of Bourgain).) Consider the set $Q = (0, 4) \times [0, 2\pi)$. Let $\phi : Q \to \mathbb{T}^d$ be given by $\phi(t, \alpha) = (r, \alpha)$ so that $\theta_1 = r\cos\alpha$ and $\theta_2 = r\sin\alpha$ satisfy

$$t = 2(\cos\theta_1 + 2\cos\theta_2)$$

Note that the segment $\{t\} \times [0, 2\pi)$ is mapped to the curve $\gamma(t)$ by $\tilde{\phi}(t, \alpha) := r(\cos\alpha, \sin\alpha)$. Note also that

$$\det \frac{\partial(r, \alpha)}{\partial(t, \alpha)} = \frac{\partial r}{\partial t} = -\frac{r}{2} \frac{1}{\theta_1 \sin\theta_1 + \theta_2 \sin\theta_2}$$

(of course, $\theta_1 = \theta_1(r, \alpha) = r\cos\alpha$, $\theta_2 = \theta_2(r, \alpha) = r\sin\alpha$). Prove that there is a constant $d_\delta > 0$ such that

$$d_\delta^{-1} < \sup_{(t,\alpha) \in [\delta, 4-\delta] \times [0, 2\pi)} \left| \det \frac{\partial(r, \alpha)}{\partial(t, \alpha)} \right| < d_\delta.$$

(c) Assume that $e \in [\delta, 4-\delta]$ (the case of negative e is handled analogously). Choose $\epsilon < \delta/2$ and decompose

$$\int_{\mathbb{T}^d} \frac{\mathrm{e}^{\mathrm{i}n\cdot\theta}}{\Phi(\theta)-z}\mathrm{d}\theta = I_1(z) + I_2(z)$$

$$I_1(z) := \int_{|\Phi(\theta)-e|\geq\epsilon} \frac{\mathrm{e}^{\mathrm{i}n\cdot\theta}}{\Phi(\theta)-z}\mathrm{d}\theta, \qquad I_2(z) := \int_{|\Phi(\theta)-e|<\epsilon} \frac{\mathrm{e}^{\mathrm{i}n\cdot\theta}}{\Phi(\theta)-z}\mathrm{d}\theta$$

where $z = e + \mathrm{i}y$. Assume that $|n_1| = |n|$. $I_1(e) := \lim_{y\downarrow 0} I_1(e+\mathrm{i}y)$ obviously exists. Integrating $I_1(e+\mathrm{i}y)$ by parts w.r.t. θ_1 and then using the change of variable near the edges of the region of integration, derive the estimate

$$|I_1(e+\mathrm{i}y)| \leq D_\delta/\epsilon|n|$$

for some constant D_δ that depends on δ only. Rewrite I_2 using the change of the variable,

$$I_2(e+\mathrm{i}y) = \int_{|t-e|<\epsilon, \alpha\in[0,2\pi)} \frac{\mathrm{e}^{\mathrm{i}n\tilde{\phi}(t,\alpha)}}{t-e-\mathrm{i}y}\left|\det\frac{\partial(r,\alpha)}{\partial(t,\alpha)}\right|\mathrm{d}t\mathrm{d}\alpha$$

Since the function $\mathrm{e}^{\mathrm{i}n\tilde{\phi}(t,\alpha)}\partial(r,\alpha)/\partial(t,\alpha)$ is continuosly differentiable (it is in fact analytic), the limit $I_2(e) := \lim_{y\downarrow 0} I_2(e+\mathrm{i}y)$ exists and

$$|I_2(e+\mathrm{i}y)| \leq F_\delta\epsilon,$$

where F_δ depends on δ only. Hence, the limit $I(e) := \lim_{y\downarrow 0} I(e+\mathrm{i}y)$ exists and

$$\sup_{y\geq 0}|I(e+\mathrm{i}y)| \leq \frac{D_\delta}{|n|\epsilon} + F_\delta\epsilon.$$

If $|n|^{-1/2} < \delta$, take $\epsilon = |n|^{-1/2}$ and derive the statement.

[11] (Continuation of [10]). The above argument gives more— since the change of variable in [10] is actually locally holomorphic in t, prove that the function $\mathbb{C}_+ \ni z \to R(n,z)$ has an analytic continuation across $(-4,0)\cup(0,4)$ (to the unphysical Riemann sheat). In particular, the boundary values $R(n,e)$ are analytic away from the critical points.

[12] The above analysis extends to all $d \geq 2$. If d is even, set

$$S_d := \{\pm 4j \,:\, 0 \leq 2j \leq d\}$$

and if d is odd,

$$S_d := \{\pm 2(2j+1) \,:\, 0 \leq 2j \leq d\}.$$

Note that $e \in S_d$ iff there exists θ on the constant energy surface $\Phi(\theta) = e$ such that $\nabla\Phi(\theta) = 0$. By mimicking the argument in [11] prove the following:
(1) The function $z \to R(n, z)$ has an analytic continuation to the second Riemann sheat across $[-2d, 2d] \setminus S_d$.
(2) Let $I_\delta \subset [-2d, 2d]$ be obtained by removing from $[-2d, 2d]$ the intervals centered at $j \in S_d$ and of length 2δ. Prove that

$$\sup_{e \in I_\delta, y \geq 0} |R(n, e + \mathrm{i}y)| \leq C_\delta \langle |n| \rangle^{-1/2}.$$

(3) Prove that

$$\sup_{e \in I_\delta \cap \{|e| > 2d-4\}, y \geq 0} |R(n, e + \mathrm{i}y)| \leq C_\delta \langle |n| \rangle^{-d/2}.$$

Remark. This could be quite messy if you are not careful. The proof of (3) requires use of the stationary phase arguments. (If you wish to learn how to estimate oscillatory integrals, an excellent reference is Chapter VIII in E. Stein: *Harmonic Analysis*, Princeton University Press 1993.) The proof of [12] can be found in W. Shaban, B. Vainberg.: Direct and inverse scattering problems for difference Schrödinger operators.

[13] (Discrete position and momentum operators). Consider the following operators on $\ell^2(\mathbb{Z}^d)$: $x_j \psi(n) = n_j \psi(n)$,

$$p_j \psi(n) = \frac{1}{2\mathrm{i}}(\psi(n + e_j) - \psi(n - e_j)),$$

where $\{e_j\}$ is the standard basis of $\mathbb{Z}^d$. x_j's and p_j's are the position and momentum operators of a free quantum particle (with mass $m = 1$) moving on the lattice $\mathbb{Z}^d$. The energy operator of this particle is $-\Delta/2$. Set $x_j(t) := \exp(-\mathrm{i}t\Delta/2) x_j \exp(\mathrm{i}t\Delta/2)$.
(1) Prove that $[p_i, x_j] = \mathrm{i}\delta_{ij}$.
(2) Prove that $[\Delta, x_j] = 2\mathrm{i}p_j$. Derive the Newton law

$$x_j(t) = x_j + tp_j.$$

(3) The Newton law for a free quantum particle on the lattice is equivalent to a well known Bessel function identity. Find this identity.

[14] (Unique continuation for the discrete Laplacian, Shaban-Vainberg). Let $\psi \in \ell^2(\mathbb{Z}^d)$ be such that $\Delta\psi(n) = E\psi(n)$ for all $|n| \geq R$.
(1) If $E \in [-2d, 2d] \setminus S_d$, prove that $\psi(n) = 0$ for $|n| \geq R$.
(2) What can you say about the case $E \in S_d$?
Suggestion: The condition $\Delta\psi(n) = E\psi(n)$ for $|n| \geq R$ implies that there are constants α_m, $|m| < R$, such that

$$\psi(n) = \sum_{|m| < R} \alpha_m R(m - n, E)$$

for all $|n| \geq B_R$.

The remaining part of this project deals with some conceptual issues concerning the definition of the discrete Laplacian.

[15] Let $Y := (X_1, \cdots, X_d)$, where X_j are independent random variables such that $P(X_j = \pm 1) = 1/2$. Let $Y_t, t \in \mathbb{N}$, be independent random variables with same distribution as X. The sequence of random variables $S_t = Y_1 + \cdots + Y_t$ is called a simple random walk on $\mathbb{Z}^d$. Denote $p(t,n) := P(S_t = n)$—this is the probability that the random walk will visit site n at time t.
(a) Let $n \in \mathbb{Z}^d$. Prove that

$$p(t,n) = \frac{1}{2d} \sum_{|m-n|=1} p(t-1,m). \tag{21}$$

Let

$$\hat{p}(t,\theta) := (2\pi)^{-d/2} \sum_{n \in \mathbb{Z}^d} p(t,n) \mathrm{e}^{-\mathrm{i}n\cdot\theta}.$$

Deduce that $\hat{p}(t,\theta) = \Phi(\theta)^t$.
(b) The simple random walk is called recurrent if $\sum_{t=0}^{\infty} p(t,0) < \infty$, otherwise it is called transient. Prove that if $\sum_t p(t,0) < \infty$, then the random walk will P-a.s. return to the origin infinitely many times. Prove that if $\sum_t p(t,0) = \infty$, then the random walk will P-a.s. return to the origin only finitely many times. Show that

$$\sum_{t=0}^{\infty} p(t,0) = 2d(2\pi)^{-d/2} \int_{\mathbb{T}^d} \frac{\mathrm{d}\theta}{2d - \Phi(\theta)},$$

and deduce that the simple random walk on $\mathbb{Z}^d$ is transient iff $d \geq 3$.
(c) Consider now the operator $H_\lambda = \Delta + \lambda(\delta_0|\cdot)\delta_0$, where $\lambda \in \mathbb{R}$. Prove that if $d = 1,2$ then H_λ has one eigenvalue outside $[-2d, 2d]$ for all $\lambda \neq 0$ and estimate the location of this eigenvalue. Show that for $d \geq 3$ there exist Λ_d such that for $|\lambda| < \Lambda_d$ H_λ has no eigenvalues outside $[-2d,2d]$ and for $|\Lambda| > \Lambda_d$ H_λ has one eigenvalue outside $[-2d,2d]$. What happens if $|\lambda| = \Lambda_d$? How does your arguments in (c) relate to the arguments in (b)?

[16] (Continuation of [15]) Let $\mathbb{Z}^d\Delta x = \{n\Delta x : n \in \mathbb{Z}^d\}$ and consider a simple random walk on this lattice with time interval Δt. This random walk is described by the random variables $S_{j\Delta t} = (Y_1 + \cdots + Y_j)\Delta x$, where $j \in \mathbb{N}$. Let $p(j\Delta t, n\Delta x)$ be the probability that

the random walk visits site $n\Delta x$ at time $j\Delta t$.
(a) Prove that

$$p(j\Delta t, n\Delta x) = \frac{1}{2d} \sum_{|m-n|=1} p((j-1)\Delta t, m\Delta x). \tag{22}$$

This equation can rewritten as

$$p(j\Delta t, n\Delta x) - p((j-1)\Delta t, n\Delta x) = \frac{1}{2d} \sum_{|m-n|=1} [p((j-1)\Delta t, m\Delta x) - p((j-1)\Delta t, n\Delta x)] \tag{23}$$

This is the discrete diffusion equation.
(b) The usual diffusion equation in $\mathbb{R}^d$ arises from (23) as follows. Let $N_0 : \mathbb{Z}^d\Delta x \to \mathbb{R}$ be the initial probability distribution of a particle conducting the random walk. Equivalently, you may think about N_0 as the initial density of a gas of particles conducting independent random walks on the lattice $\mathbb{Z}^d\Delta x$. The expected number of particles at $k\Delta x$ at time $j\Delta t$ is

$$N_{j\Delta t}(k\Delta x) = \sum_{m\in\mathbb{Z}^d} N_0(m\Delta x)p(j\Delta t, (k-m)\Delta x).$$

If f is a function on $\mathbb{Z}\Delta x$, set

$$\hat{f}(\theta) = \left(\frac{\Delta x}{2\pi}\right)^{d/2} \sum_{m\in\mathbb{Z}^d} f(m\Delta x)\mathrm{e}^{-\mathrm{i}m\theta\Delta x}, \qquad \theta \in [-\pi/\Delta x, \pi/\Delta x)^d.$$

Prove that

$$\hat{N}_{j\Delta t}(\theta) = \left(\frac{\Delta x}{2\pi}\right)^{d/2} \Phi(\theta\Delta x)^j.$$

Notice that

$$(\Delta x)^{d/2}\hat{N}_0(\theta) = (2\pi)^{-d/2} \sum_{m\in\mathbb{Z}^d} N_0(m\Delta x)\mathrm{e}^{-\mathrm{i}m\theta\Delta x}(\Delta x)^d.$$

Assume that there exist a function (or distribution) $N_0(x)$ on $\mathbb{R}^d$ such that

$$\lim_{\Delta x\to 0} (\Delta x)^{d/2}\hat{N}_0(\theta) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} N_0(x)\mathrm{e}^{-\mathrm{i}x\theta}\mathrm{d}x. \tag{24}$$

If k is chosen so that $k\Delta x \to x$ as $\Delta x \to 0$, and j is chosen so that $j\Delta t \to t$ as $\Delta t \to 0$, and the limits are taken so that $D = (\Delta x)^2/2\Delta t$ is kept fixed, we obtain that

$$N_t(x) := \lim N_{j\Delta t}(k\Delta x) = \frac{1}{(2\pi tD)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|x-y|^2}{4Dt}} N_0(y)\mathrm{d}y. \tag{25}$$

Note that

$$(\partial_t - \sum_j \partial_{x_j}^2) N_t(x) = 0, \tag{26}$$

and so N_t satisfies the diffusion equation in $\mathbb{R}^d$ with the diffusion constant D. If $N_0(k\Delta x) = 1$ for $k = 0$ and 0 otherwise (so $N_0(y)$ is the delta function at the origin), then

$$N_t(x) = \frac{1}{(2\pi t D)^{d/2}} \mathrm{e}^{-\frac{|x|^2}{4Dt}}$$

In this way, starting with the simple random walk and discrete Laplacian on derives the diffusion equation in $\mathbb{R}^d$. Justify mathematically the steps (24) and (25) by making appropriate assumptions.

Remark. According to [16] and (23) the operator

$$\tilde{\Delta}\psi(n) = \frac{1}{2d} \sum_{|m-n|=1} (\psi(m) - \psi(n))$$

is the discrete analog of the Laplace operator $\sum_j \partial_{x_j}^2$. However, it has became a custom to work instead with $\Delta = 2d\tilde{\Delta} + 2d\mathbf{1}$ and call this operator the discrete Laplacian. This widely adopted convention is harmless and notationaly convenient.

[17] We now discuss a general random walk on $\mathbb{Z}^d$. Let Y be a random variables with values in $\mathbb{Z}^d$ and let $p(n) := P(Y = n)$, $n \in \mathbb{Z}^d$. Clearly, $p(n) \geq 0, \sum_n p(n) = 1$. Let $Y_t, t \in \mathbb{N}$ be i.i.d. random variables with same distribution as Y and let $S_t = Y_1 + \cdots + Y_t$. Let $p(t, n) = P(S_t = n)$ (so $p(1, n) = p(n)$).
(a) Prove that

$$p(t+1, n) = \sum_{m \in \mathbb{Z}^d} p(t, m) p(n - m).$$

(b) For $\theta \in [0, 2\pi)^d$ set

$$\hat{p}(t, \theta) = (2\pi)^{-d/2} \sum_{n \in \mathbb{Z}^d} p(t, n) \mathrm{e}^{-\mathrm{i}n\theta}.$$

We write $\hat{p}(\theta) := p(1, \theta)$. Prove that

$$\hat{p}(t, \theta) = \hat{p}(\theta)^t, \qquad p(t, n) = (2\pi)^{-d/2} \int_{\mathbb{T}^d} \hat{p}(\theta)^t \mathrm{e}^{\mathrm{i}n\theta} \mathrm{d}\theta.$$

(c) The random walk S_n is called reccurent if $\sum_t p(t, 0) < \infty$ and transient if $\sum_t p(t, 0) = \infty$. Prove that if $\sum_t p(t, 0) < \infty$, then the random walk will P-a.s. return to the origin infinitely many times. Prove that if $\sum_t p(t,0) = \infty$, then the random walk will P-a.s. return to the origin only finitely many times. Show that

$$\sum_{t=0}^{\infty} p(t,0) = (2\pi)^{-d/2} \int_{\mathbb{T}^d} \frac{\mathrm{d}\theta}{1-\hat{p}(\theta)},$$

(d) Consider a random work on $\mathbb{Z}$ with $p(n) = c(1+|n|)^{-\alpha}$, $c^{-1} = \sum(1+|n|)^{-\alpha}$. Prove that this random walk is recurrent if $\alpha \geq 2$ and transient if $1 < \alpha < 2$.
(e) Consider the random walk on $\mathbb{Z}$ with $p(1) = p$, $p(-1) = q$, $p+q = 1$. Prove that this random walk is reccurent iff $p = q = 1/2$.
(f) The Laplacian associated to the random walk S_n is

$$\tilde{\Delta}\psi(n) = \sum p(n-m)\psi(m)$$

In the Fourier representation, $\tilde{\Delta}$ acts as operator of multiplication by $\hat{p}(\theta)$ and $\mathrm{sp}(\tilde{\Delta}) = [\min \hat{p}(\theta), \max \hat{p}(\theta)]$.
(g) Study the spectrum of $\tilde{\Delta}$.
(e) Consider the operator $H_\lambda = \tilde{\Delta} + \lambda(\delta_0|\cdot)\delta_0$, where $\lambda \in \mathbb{R}$. Show that for all λ H_λ can have at most one eigenvalue outside $\mathrm{sp}(\tilde{\Delta})$ Prove that if the random walk is reccurent, then H_λ has one eigenvalue outside $\mathrm{sp}(\tilde{\Delta})$ for all $\lambda \neq 0$ and estimate the location of this eigenvalue. Prove that if the random walk is transient, then there exists an interval $[\Lambda_1, \Lambda_2]$ such that for $\lambda \in (\Lambda_1, \Lambda_2)$ H_λ has no eigenvalues outside $\mathrm{sp}(\tilde{\Delta})$ and for $\lambda \notin [\Lambda_1, \Lambda_2]$, H_λ has an eigenvalue outside $\mathrm{sp}(\tilde{\Delta})$. Compute Λ_1, Λ_2. What happens if $\lambda = \Lambda_1, \Lambda_2$?
(h) Notice that $p(0) = 1$. Assume that $0 \in \mathrm{int}\, \mathrm{sp}(\tilde{\Delta})$, that $\hat{p}(\theta)$ is twice differentiable at 0 and that $\nabla\hat{p}(0) = 0$. If $\mathrm{Hess}\, \hat{p}(0) = \gamma\mathbf{1}$, following the problem [16] prove that the scaling limit of the discrete difussion equation associated to S_n is again the standard diffusion equation (26) with the diffusion constant $D\gamma$. Compute the scaling limit if $\mathrm{Hess}\, \hat{p}(0) > 0$. Interpret your result geometrically. What is the scaling limit if $\nabla\hat{p}(0) \neq 0$?

The previous problem suggests that from the physical point of view there is a considerable freedom in the choice of the discrete analog of the continuous Laplacian. (This fact is well known in the physics literature—in the mathematics literature, however, virtually all papers deal with the standard discrete Laplacian Δ). Although seemingly a simplest choice, from a certain point of view Δ is also a pathological choice. In particular, since the surfaces $\Phi(\theta) = E$ are not convex for $|E| \leq 2d-4$ (and in fact have a rather complicated structure), the resolvent of Δ for energies $|E| \leq 2d-4$ has a rather unusual behavior. This behavior has a considerable impact on the scattering properties of the discrete Schrödinger operators $-\Delta + V$ (it is an interesting question whether this pathological behavior has been already observed in the physics literature).

Molchanov and Vainberg have suggested an alternative to the standard Laplacian. Consider a random walk on $\mathbb{Z}^d$ given by $p(n) = 2^{-d}$ if $n_j = \pm 1$ for all j (draw a picture—the random walk jumps along the diagonals). The corresponding discrete Laplacian is

$$\Delta_{\mathrm{MV}}\psi(n) = 2^{-d} \sum_{|n-m|_+=1} \psi(m),$$

where $|n|_+ = (\sum n_j^2)^{1/2}/d^{1/2}$. We will call this operator the Molchanov-Vainberg Laplacian. In the Fourier representation Δ_{MV} acts as operator of multiplication by $\Phi_{\mathrm{MV}}(\theta) := \prod_j \cos\theta_j$.

[18] Prove that for $d = 2$, $4\Delta_{\mathrm{MV}}$ is unitarily equivalent to Δ.

[19] For $d = 3$, using Mathematica or Maple, draw the surfaces $\Phi_{\mathrm{MV}}(\theta) = j/10$, $j = 1, \cdots, 9$.

[20] Molchanov and Vainberg have conjectured that the surfaces $\Phi_{\mathrm{MV}}(\theta) = E$ are strictly convex for $E \in (-1, 0) \cup (0, 1)$. This conjecture has been recently proven by Philippe Poulin.

[21] Repeat the first part of the project for the operator Δ_{MV}.

Project III: Discrete Schrödinger operators

Let $V : \mathbb{Z}^d \to \mathbb{R}$ be a given function (potential). We denote by the same letter the corresponding multiplication operator on $\ell^2(\mathbb{Z}^d)$, $(V\psi)(n) = V(n)\psi(n)$. The corresponding discrete Schrödinger operator is defined by

$$H = -\Delta + V$$

where Δ is the discrete Laplacian. H is a selfadjoint operator on the Hilbert space $\ell^2(\mathbb{Z}^d)$ and $\mathrm{Dom}(H) = \mathrm{Dom}(V)$. In the lattice approximation, H is the energy operator of non-relativistic quantum particle moving under the influence of the potential V.

In this project we will study H. To simplify the notation we set $H_0 := -\Delta$. If $\Lambda \subset \mathbb{Z}^d$, $C_{\mathrm{fin}}(\Lambda)$ denotes the set of function with compact support contained in Λ.

[1] Prove that $\mathrm{sp}_{\mathrm{ess}}(H) = \emptyset$ if and only if $\lim_{|n|\to\infty} |V(n)| = \infty$.

[2] Denote $B_R := \{|n| \leq R\}$ in $\mathbb{Z}^d$, $B_{\overline{R}} = \mathbb{Z}^d \setminus B_R$.
(1) Prove that

$$\inf \mathrm{sp}_{\mathrm{ess}}(H) = \sup_{R>0} \inf_{\psi \in C_{\mathrm{fin}}(B_{\overline{R}}), \|\psi\|=1} (\psi|H\psi).$$

This is the discrete analog of the Persson theorem.
(2) Prove that $e \in \mathrm{sp}_{\mathrm{ess}}(H)$ iff there exist a sequence of unit vectors $\psi_j \in C_{\mathrm{fin}}(B_{\bar{j}})$ such that $\lim_{j\to\infty} \|(H-e)\psi_j\| = 0$.
(3) Prove that $\lim_{|n|\to\infty} |V(n)| = 0 \;\Rightarrow\; \mathrm{sp}_{\mathrm{ess}}(H) = \mathrm{sp}_{\mathrm{ess}}(H_0)$.
(4) Construct V such that $\limsup_{|n|\to\infty} |V(n)| = \infty$ and $\mathrm{sp}_{\mathrm{ess}}(H) = \mathrm{sp}_{\mathrm{ess}}(H_0)$.

[4] Assume that $V(n) = O(\langle |n| \rangle^{-1-\epsilon})$ for some $\epsilon > 0$ (such potentials are called short range). Prove that the wave operators

$$W^{\pm} := \mathrm{s} - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0}$$

exist. In particular, $[-2d, 2d] \subset \mathrm{sp}_{\mathrm{ac}}(H)$.
Hint: (a) We will focus on W^+. To prove the existence of W^+ it suffices to show that there exists $D \subset \ell^2(\mathbb{Z}^d)$ such that:
(a1) The linear span of D is dense in $\ell^2(\mathbb{Z}^d)$.
(a2) For $\psi \in D$,

$$\int_0^\infty \|V\mathrm{e}^{-\mathrm{i}tH_0}\psi\|\mathrm{d}t < \infty.$$

This sufficient condition is called Cook's criterion. Prove Cook's criterion.
(b) Let D be a set of functions $\hat{\psi}$ such that $\hat{\psi}(\theta) \in C^\infty(\mathbb{T}^d)$ and the points whose all coordinates are $k\pi/2$, $k = 0,1,2,3$, are not in $\mathrm{supp}\,\hat{\psi}$. Prove that (a1) holds for D. If $\psi \in D$, prove the following estimates:
(b1) There exists $\gamma_1 > 0$ such that all $k \in \mathbb{N}$

$$\left| \int_{\mathbb{T}^d} \mathrm{e}^{\mathrm{i}t\Phi(\theta)+\mathrm{i}n\cdot\theta}\hat{\psi}(\theta)\mathrm{d}\theta \right| \le c_k \langle |n| \rangle^{-k}.$$

whenever $|n| \ge \gamma_1 t$. The constant c_k depends on k only.
(b2) There exists $\gamma_2 > 0$ such that for all $k \in \mathbb{N}$,

$$\left| \int_{\mathbb{T}^d} \mathrm{e}^{\mathrm{i}t\Phi(\theta)+\mathrm{i}n\cdot\theta}\hat{\psi}(\theta)\mathrm{d}\theta \right| \le d_k \langle t \rangle^{-k}.$$

whenever $|n| \le \gamma_2 t$. The constant d_k depends on k only.
(b3) There exist constant C such that if $\gamma_2 t \le |n| \le \gamma_1 t$, then

$$\left| \int_{\mathbb{T}^d} \mathrm{e}^{\mathrm{i}t\Phi(\theta)+\mathrm{i}n\cdot\theta}\hat{\psi}(\theta)\mathrm{d}\theta \right| \le C \langle t \rangle^{-d/2}.$$

(c) For $\psi \in \mathcal{D}$, estimate $\|Ve^{-\mathrm{i}tH_0}\psi\|$ using

$$\|Ve^{-\mathrm{i}tH_0}\psi\|^2 \le C \sum_{n\in\mathbb{Z}^d} |n|^{-2-2\epsilon} \left| \int_{\mathbb{T}^d} \mathrm{e}^{\mathrm{i}t\Phi(\theta)+\mathrm{i}n\cdot\theta}\hat{\psi}(\theta)\mathrm{d}\theta \right|^2$$

and the estimates in (b) (you should split the sum into three parts, $|n| \leq \gamma_1 t$, $\gamma_1 t < |n| < \gamma_2 t$, $|n| \geq \gamma_2 t$). Conclude that $\|Ve^{-\mathrm{i}tH_0}\psi\| = O(\langle t\rangle^{-1-\epsilon})$ and hence that Cook's criterion holds.

[5] Let V be short range. To the best of my knowledge, the following questions are open problems:
(1) Can H have embedded eigenvalues in $(-2d, 2d)$ (or $[-2d, 2d] \setminus S_d$)? (In the continuous case, the answer is negative, see Theorem XIII.58 in [RS4]).
(2) Please read the statement of the Agmon-Kato-Kuroda theorem— Theorem XIII.33 in [RS4]. Does this result holds in the discrete case?
(3) Does the relation $\mathrm{Ran}W^+ = \mathrm{Ran}W^-$ always holds? (In the continuous case the answer is negative—see the Pearson example in [RS3], Section XI.4).
Remark. If $V \in \mathrm{C}_{\mathrm{fin}}(\mathbb{Z}^d)$, then (1) follows from [14] in Project II.

[6] Assume that $V(n) = O(\langle n\rangle^\alpha)$ for some $\alpha \in \mathbb{R}$. Let

$$c(r) = \frac{\#\{n \in B_r \,:\, V(n) \neq 0\}}{r^d}.$$

Prove that if

$$\int_1^\infty r^\alpha \sqrt{c(r)}\mathrm{d}r < \infty,$$

then the wave operators $W^\pm$ exist. (This result is due to Molchanov and Vainberg. Follow the same steps as in the problem 4).

[7] Assume that H has only point spectrum. Denote by x the operator of multiplication by $|n|$. Simon proved that

$$\lim_{t\to\infty} t^{-2}\|x\mathrm{e}^{-\mathrm{i}tH}\delta_0\|^2 = 0. \tag{27}$$

(Simon B.: Absence of ballistic motion, Commun. Math. Phys. **134**, 209-212 (1990)). Please read this short and very instructive paper. Remarkably, this is an optimal result: del Rio-Jitomirskaya-Last-Simon have constructed examples of discrete Schrödinger operators with pure point spectrum and exponentially decaying eigenfunctions such that

$$\limsup_{t\to\infty} f(t)t^{-2}\|x\mathrm{e}^{-\mathrm{i}tH}\delta_0\|^2 = \infty$$

for any monotone f with $f(\infty) = \infty$.

In [8] below we will describe a discrete version of the Combes-Thomas argument. We will return to this argument in Lecture 10. The [8] below is the problem III.6.8. in [CL].

[8] For each $\theta \in \mathbb{C}^d$ denote by P_θ the diagonal operator

$$(P_\theta \psi)(n) = \mathrm{e}^{\mathrm{i}\theta \cdot n} \psi(n).$$

(1) Prove that

$$P_{-\theta} H P_\theta = H + Q_\theta, \qquad P_{-\theta}(H - z)^{-1} P_\theta = (H + Q_\theta - z)^{-1},$$

where Q_θ is a bounded operator independent of the potential and satisfying

$$\|Q_\theta\| \leq 2d(\mathrm{e}^{|\theta|} - 1).$$

(2) Let $\delta = \mathrm{dist}(z, \mathrm{sp}(H))$. Prove that for $|\theta| < a(\delta) := \log(1 + (4d)^{-1}\delta)$ one has

$$|(\delta_n | P_{-\theta}(H - z)^{-1} P_\theta \delta_m)| = \left|\mathrm{e}^{\mathrm{i}\theta(m-n)} (\delta_n | (H - z)^{-1} \delta_m)\right| \leq \frac{2}{\delta}.$$

(3) Deduce that

$$|(\delta_n | (H - z)^{-1} \delta_m)| \leq \frac{2}{\delta} \mathrm{e}^{-a(\delta)|n-m|}.$$

[9] Let e be an isolated eigenvalue of H.
(1) Prove that

$$|(\delta_n | \mathbf{1}_e(H) \delta_m)| \leq \mathrm{e}^{-a(\delta/2)|n-m|},$$

where $\delta = \mathrm{dist}(e, \mathrm{sp}(H) \setminus \{e\})$ and $a(\delta)$ is as in [8].
(Hint: Use that

$$\mathbf{1}_e(H) = \frac{1}{2\pi \mathrm{i}} \int_{|z-e|=\delta/2} (z - H)^{-1} \mathrm{d}z,$$

and the problem [8].)
(2) Assume that e is a simple eigenvalue and let ψ be the corresponding eigenfunction. Prove that there exist constants $C > 0$ and $D > 0$ such that

$$|\psi(n)| \leq C\mathrm{e}^{-D|n|}.$$

What can you say about C and D?
(3) Extend (2) to the case where e is an eigenvalue of finite multiplicity.

[10] A discrete Schrödinger operator may have isolated eigenvalues of infinite multiplicity.
(1) Let $\lambda \in \mathbb{R}$ be such that $H_0 + \lambda(\delta_0 | \cdot)\delta_0$ has an eigenvalue $e(\lambda)$ outside $\mathrm{sp}(H_0) = [-2d, 2d]$. Let $\Gamma = \{n_j\}_{j \in \mathbb{N}}$ be a set of sites in $\mathbb{Z}^d$ and let

$$V(n) = \begin{cases} \lambda & n \in \Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Let $d_j := \mathrm{dist}(n_j, \Gamma \setminus \{n_j\}) > \mathrm{e}^j$. If $d_j > e^j$, prove that $\mathrm{sp}_{\mathrm{ess}}(H) = \mathrm{sp}(H_0) \cup \{e(\lambda)\}$.
(2) Can you prove (1) under the assumption $d_j \to \infty$?
(3) What can you say about the decay of eigenfunctions in $\mathrm{Ran}\mathbf{1}_{e(\lambda)}(H)$?
(This problem is a special case of the Molchanov-Vainberg theorem).

[11] (Continuation of the problem [10]). Assume that $d \geq 2$ and that $d_j > e^j$,
(1) Prove that the wave operators

$$W^{\pm} = \mathrm{s} - \lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH} \mathrm{e}^{-\mathrm{i}tH_0} \tag{28}$$

exists. Hence $\mathrm{sp}_{\mathrm{ac}}(H) = \mathrm{sp}_{\mathrm{ac}}(H_0)$.
(2) Prove that the spectrum of H in $[-2d, 2d] \setminus S_d$ is purely absolutely continuous.
(3) Prove that the wave operators $W^{\pm}$ are complete on the spectral interval $[-2d, 2d] \setminus S_d$.
(4) How much you can relax the assumption $d_j > e^j$?
(Hint: This a bit hard–you might need to read a couple of papers before attempting this problem. Done right and suitably generalized this problem may lead to a publishable note).

[12] (Continuation of [11]) If $d = 1$, then [11] is false. It follows from the Kiselev-Last-Simon theorem that if $\lim_{j\to\infty} n_j/n_{j+1} \to 0$, then the spectrum of H in $(-2, 2)$ is purely singular continuous (under the above assumptions $[-2, 2] = \mathrm{sp}_{\mathrm{sc}}(H)$).

[13] The following result is known as Sch'nol's theorem. Let $\psi : \mathbb{Z}^d \to \mathbb{R}$ be polynomially bounded.
(1) If $H\psi = e\psi$, then $e \in \mathrm{sp}(H)$.
(2) If $H\psi = e\psi$ and $\psi \notin \ell^2(\mathbb{Z}^d)$, then $e \in \mathrm{sp}_{\mathrm{ess}}(H)$.
Prove this result by mimicking the proof of Theorem 2.9 in [CFKS] (they treat the continuos case—the discrete case is much easier).

[14] The following result is known as Sch'nol-Berezanskii-Simon theorem. Let μ be the spectral measure for H and $\epsilon > 0$. Then for μ-a.e. $e \in \mathbb{R}$ there is a solution of the equation $H\psi = e\psi$ such that

$$|\psi(n)| \leq C(e, \epsilon)(1 + |n|)^{d/2+\epsilon}.$$

Combined with [13] this result yields that $\mathrm{sp}(H)$ is the closure of the set of all e for which $H\psi = e\psi$ has a polynomially bounded solution.