MATH 743
SPECTRAL THEORY OF ANDERSON TYPE HAMILTONIANS
Instructor: Vojkan Jaksic
LECTURE 5

Random Selfadjoint Operators

O.1 References

[CL] Carmona R., Lacroix J.: *Spectral Theory of Random Schrödinger Operators*, Birkhäuser, 1990.

[PF] Pastur L., Figotin A.: *Spectra of Random and Almost-Periodic Operators*, Springer-Verlag 1992.

[CFKS] Cycon H.L., Froese R.G., Kirsch W., Simon B.: *Schrödinger Operators*, Springer-Verlag 1987.

1.1. Introduction

In this lecture we will start with the main topic of the course. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{H}$ a separable Hilbert space. Suppose that we are given a function

$$\Omega \ni \omega \to H_\omega \tag{1}$$

with values in the set of all linear operators on $\mathcal{H}$. We will be primarily interested in the case where H_ω is selfadjoint or bounded. Our goal is to study spectral properties of H_ω which hold with probability one. Compared with the traditional approach described in the Lecture 3, the random approach yields a radically different insight into the spectral theory.

In this lecture we will focus on the measurability properties of the function (1). In fact for the purpose of this lecture the probabilistic framework is irrelevent and we will deal with operator-valued functions on an arbitrary measuruable space.

In the research literature the questions of measurability are often left as an exercise to the reader or altgether skipped, and indeed in most cases the necessary measurability properties are relatively easy to establish. In the latter lectures we will often follow the spirit of the research literature, and whenever the author deems them obvious, the measurability questions will be left to the reader.

1.2. Basic notions

Let $(M, \mathcal{F})$ be a measurable space and X a topological space. A function $f : M \to X$ is called measurable if the inverse image of every open set is in $\mathcal{F}$.

A function between two topological spaces is called continuous if the inverse image of every open set is open.

Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{B}(\mathcal{H})$ the vector space of all bounded operators on $\mathcal{H}$. We distinguish three topologies in $\mathcal{B}(\mathcal{H})$:
(1) The uniform topology is induced by the operator norm on $\mathcal{B}(\mathcal{H})$.
(2) The strong topology is the minimal topology w.r.t. which the functions $\mathcal{B}(\mathcal{H}) \ni A \to A\psi \in \mathcal{H}$ are continuous for all $\psi \in \mathcal{H}$.
(3) The weak topology is the minimal topology w.r.t. which the functions $\mathcal{B}(\mathcal{H}) \ni A \to (\phi|A\psi) \in \mathbb{C}$ are continuous for all $\phi, \psi \in \mathcal{H}$.

The weak topology is finer then the strong topology, and the strong topology is finer then the uniform topology.

A function $f : M \to \mathcal{B}(\mathcal{H})$ is uniform/strong/weak measurable if it is measurable with respect to the uniform/strong/weak topology. Obviously, uniform measurability $\Rightarrow$ strong measurability $\Rightarrow$ weak measurability. Note that f is weakly measurable iff the function $M \ni \omega \to (\phi|f(\omega)\psi) \in \mathbb{C}$ is measurable for all $\phi, \psi \in \mathcal{H}$.

Theorem 1.2.1. A function $f : M \to \mathcal{B}(\mathcal{H})$ is uniform measurable iff it is weakly measurable.

Proof. Assume that f is weakly measurable. Let $B(r) := \{A \in \mathcal{B}(\mathcal{H}) : \|A\| \leq r\}$. We need to show that

$$\{\omega \in M \ : \ f(\omega) \in B(r)\} \tag{2}$$

is measurable. Let $\{\phi_n\}$ be a countable dense set on the sphere $\{\psi \in \mathcal{H} : \|\psi\| = 1\}$. Then for all ω,

$$f(\omega) \in B(r) \qquad \Longleftrightarrow \qquad \sup_n \|f(\omega)\phi_n\| \leq r.$$

Since f is weakly measurable, the function $\omega \to \|f(\omega)\phi_n\|$ is measurable for all n, and so the set (2) is measurable. $\square$

A function $f : M \to \mathcal{B}(\mathcal{H})$ is *measurable* iff it is weakly measurable (which is equivalent to requiring that f is strongly or uniform measurable).

Definition 1.2.2. Let $\omega \to H_\omega$ be a function with values in (possibly unbounded) selfadjoint operators on $\mathcal{H}$. We say that H_ω is *selfadjoint measurable* if for all $z \in \mathbb{C} \setminus \mathbb{R}$ the function

$$\omega \to (H_\omega - z)^{-1} \in \mathcal{B}(\mathcal{H})$$

is measurable.

Theorem 1.2.3. Assume that $\omega \to H_\omega$ is selfadjoint measurable and that $h \in C_{\mathrm{b}}(\mathbb{R})$. Then the function $\omega \to h(H_\omega)$ is measurable.

Proof. Let $\mathcal{T} \subset C_{\rm b}(\mathbb{R})$ be the class of functions such that $\omega \to h(H_\omega)$ is measurable By definition, $(x-z)^{-1} \in \mathcal{T}$ for all $z \in \mathbb{C}\setminus\mathbb{R}$. Since the linear span of $\{(x-z)^{-1} : z \in \mathbb{C}\setminus\mathbb{R}\}$ is dense in the Banach space $C_0(\mathbb{R})$, $C_0(\mathbb{R}) \subset \mathcal{T}$. Note also that if $h_n \in \mathcal{T}$, $\sup_{n,x}|h_n(x)| < \infty$, and $h_n(x) \to h(x)$ for *all* x, then $h \in \mathcal{T}$. Hence, $\mathcal{T} = C_{\rm b}(\mathbb{R})$. □

From this theorem it follows that the functions $\omega \to \mathbf{1}_B(H_\omega)$ (B Borel) and $\omega \to \exp(\mathrm{i}tH_\omega)$ are measurable. One can also easily show that if $h : \mathbb{R} \to \mathbb{R}$ is in arbitrary real valued function, then $\omega \to h(H_\omega)$ is selfadjoint measurable.

1.3. Measurability of projections and spectral measures.

Throughout this and the next subsection $\omega \to H_\omega$ is given selfadjoint measurable function.

Proposition 1.3.1. The function $\omega \to \mathbf{1}_{\rm cont}(H_\omega)$ is measurable.

Proof. Let $\{\phi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$ and let P_n be the orthogonal projection on the subspace spanned by $\{\phi_k\}_{k\geq n}$. The RAGE theorem yields that for $\varphi, \psi \in \mathcal{H}$,

$$(\varphi|\mathbf{1}_{\rm cont}(H_\omega)\psi) = \lim_{n\to\infty}\lim_{T\to\infty}\frac{1}{T}\int_0^T(\varphi|\mathrm{e}^{\mathrm{i}tH_\omega}P_n\mathrm{e}^{-\mathrm{i}tH_\omega}\psi)\mathrm{d}t \tag{3}$$

(the proof of (3) is a homework problem), and the statement follows. □

Proposition 1.3.2. The function $\omega \to \mathbf{1}_{\rm ac}(H_\omega)$ is measurable.

Proof. By Theorem 2 in Lecture 2, for all $\psi \in \mathcal{H}$,

$$(\psi|\mathbf{1}_{\rm ac}(H_\omega)\psi) = \lim_{M\to\infty}\lim_{p\uparrow 1}\lim_{\epsilon\downarrow 0}\frac{1}{\pi^p}\int_{-M}^{M}\left[\mathrm{Im}(\psi|(H_\omega - x - \mathrm{i}\epsilon)^{-1}\psi)\right]^p\mathrm{d}x,$$

and so $\omega \to (\psi|\mathbf{1}_{\rm ac}(H_\omega)\psi)$ is measurable. This fact and the polarization identity yield the statement. □

Proposition 1.3.3. The functions $\omega \to \mathbf{1}_{\rm sc}(H_\omega)$ and $\omega \to \mathbf{1}_{\rm pp}(H_\omega)$ are measurable.

Proof. $\mathbf{1}_{\rm sc}(H_\omega) = \mathbf{1}_{\rm cont}(H_\omega) - \mathbf{1}_{\rm ac}(H_\omega)$ and $\mathbf{1}_{\rm pp}(H_\omega) = \mathbf{1} - \mathbf{1}_{\rm cont}(H_\omega)$. □

We now turn to the measurability of spectral measures.

Let $M(\mathbb{R})$ be the Banach space of all complex Borel measures on $\mathbb{R}$ (the dual of $C_0(\mathbb{R})$). A map $\omega \to \mu^\omega \in M(\mathbb{R})$ is called measurable iff for all $f \in C_{\rm b}(\mathbb{R})$ the map $\omega \to \mu^\omega(f)$ is measurable.

We denote by μ_ψ^ω the spectral measure for H_ω and ψ.

Proposition 1.3.4. The functions $\omega \to \mu_{\psi,\rm ac}^\omega$, $\omega \to \mu_{\psi,\rm sc}^\omega$, $\omega \to \mu_{\psi,\rm pp}^\omega$ are measurable.

Proof. Since for any Borel set B, $(\mathbf{1}_B(H_\omega)\psi|\mathbf{1}_{\rm ac}(H_\omega)\psi) = \mu^\omega_{\psi,\rm ac}(B)$, $(\mathbf{1}_B(H_\omega)\psi|\mathbf{1}_{\rm sc}(H_\omega)\psi) = \mu^\omega_{\psi,\rm sc}(B)$, $(\mathbf{1}_B(H_\omega)\psi|\mathbf{1}_{\rm pp}(H_\omega)\psi) = \mu^\omega_{\psi,\rm pp}(B)$, the statement follows from Propositions 1.3.2 and 1.3.3. □

Let $\{\psi_n\}$ be a cyclic set for H_ω and let $a_n > 0$ be such that $\sum_n a_n\|\psi_n\|^2 < \infty$. We denote by

$$\mu^\omega = \sum_n a_n \mu^\omega_{\psi_n},$$

the corresponding spectral measure for H_ω. Proposition 1.3.4 yields

Proposition 1.3.5. The functions $\omega \to \mu^\omega_{\rm ac}$, $\omega \to \mu^\omega_{\rm sc}$, $\omega \to \mu^\omega_{\rm pp}$ are measurable.

Let $\Sigma^{\rm ess,\omega}_{\rm ac}$ be the essential support of the a.c. spectrum of H_ω. The map

$$\omega \to (1+x^2)^{-1}\chi_{\Sigma^{\rm ess,\omega}_{\rm ac}}(x) \in L^1(\mathbb{R}, {\rm d}x) \tag{4}$$

does not depend on the choice of representative in $\Sigma^{\rm ess,\omega}_{\rm ac}$.

Proposition 1.3.6. The function (4) is weakly measurable, namely for all $h \in L^\infty(\mathbb{R}, {\rm d}x)$, the function

$$\omega \to \int_{\mathbb{R}} h(x)(1+x^2)^{-1}\chi_{\Sigma^{\rm ess,\omega}_{\rm ac}}(x){\rm d}x$$

is measurable.

Proof. It suffices to prove the statement for $h(x) = (1+x)^2\chi_A(x)$ where A is a bounded interval. Let μ^ω be a spectral measure for H_ω. By the dominated convergence theorem

$$\int_A \chi_{\Sigma^{\rm ess,\omega}_{\rm ac}}(x){\rm d}x = 2\lim_{\epsilon\downarrow 0}\lim_{\delta\downarrow 0}\int_A \frac{P_{\mu^\omega_{\rm ac}}(x+{\rm i}\delta)}{P_{\mu^\omega_{\rm ac}}(x+{\rm i}\epsilon)+P_{\mu^\omega_{\rm ac}}(x+{\rm i}\delta)}{\rm d}x, \tag{5}$$

and the statement follows. □

1.4. Measurability and cyclic subspaces

Let $\mathcal{H}_{\omega,\psi}$ be the cyclic subspace spanned by H_ω and ψ. We denote by $\mathbf{1}_{\omega,\psi}$ be the spectral projection on $\mathcal{H}_{\omega,\psi}$.

Proposition 1.4.1. The function $\omega \to \mathbf{1}_{\omega,\psi}$ is measurable.

Proof. Let $\{z_k\}$ be an ordering of points in $\mathbb{C}\setminus\mathbb{R}$ whose both coordinates are rational numbers. Let $\phi_{k,\omega} = (H_\omega - z_k)^{-1}\psi$. Assume that $\psi \neq 0$ (otherwise the statement is obvious). Then, $\phi_{k,\omega} \neq 0$, the set $\{\phi_{k,\omega}\}$ is total in $\mathcal{H}_{\omega,\psi}$ for all ω, and the functions

$\omega \to \phi_{k,\omega} \in \mathcal{H}$ are measurable. Let $\{u_{k,\omega}\}$ be an orthonormal basis of $\mathcal{H}_{\omega,\psi}$ obtained from $\{\phi_{k,\omega}\}$ by the Gram-Schmidt process. Then the functions $\omega \to u_{k,\omega}$ are measurable. Since for any $\phi \in \mathcal{H}$,

$$\mathbf{1}_{\omega,\psi}\phi = \sum_k (u_{k,\omega}|\phi)u_{k,\omega},$$

the function $\omega \to \mathbf{1}_{\omega,\psi}$ is measurable. □

Proposition 1.4.2. Let ψ, φ be two vectors in $\mathcal{H}$. Then the sets

$$M_1 := \{\omega : \mathcal{H}_{\omega,\psi} \perp \mathcal{H}_{\omega,\varphi}\}, \qquad M_2 := \{\omega : \mathcal{H}_{\omega,\psi,\mathrm{sing}} = \mathcal{H}_{\omega,\varphi,\mathrm{sing}}\},$$

are measurable.

Proof. Let $\{\phi_i\}$ be a countable dense set in $\mathcal{H}$. Then

$$M_1 = \{\omega \,:\, (\mathbf{1}_{\omega,\psi}\phi_i|\mathbf{1}_{\omega,\varphi}\phi_j) = 0, \ \ \forall i,j\},$$

and the set on the rhs is measurable by the previous proposition.

Let $\mathbf{1}_{\omega,\psi,\mathrm{sing}}$ be the projection on $\mathcal{H}_{\omega,\psi,\mathrm{sing}}$. Then, $\mathbf{1}_{\omega,\psi,\mathrm{sing}} = \mathbf{1}_{\mathrm{sing}}(H_\omega)\mathbf{1}_{\omega,\psi}$ and so the function $\omega \to \mathbf{1}_{\omega,\psi,\mathrm{sing}}$ is measurable. The measurability of M_2 follows from

$$M_2 = \{\omega : (\phi_i|\mathbf{1}_{\omega,\psi,\mathrm{sing}}\phi_j) = (\phi_i|\mathbf{1}_{\omega,\varphi,\mathrm{sing}}\phi_j), \ \ \forall i,j\}.$$

□

1.5. Homework.

[1] Let $M \ni \omega \to H_\omega$ be a function with values in selfadjoint operators on $\mathcal{H}$. Prove that the following statements are equivalent:
(a) $\omega \to (H_\omega - z)^{-1}$ is measurable for all $z \in \mathbb{C} \setminus \mathbb{R}$.
(b) $\omega \to \exp(\mathrm{i}tH_\omega)$ is measurable for all $t \in \mathbb{R}$.
(c) $\omega \to \mathbf{1}_B(H_\omega)$ is measurable for all Borel sets B.

[2] Prove Relation (3).

[3] In the literature, the proof of the measurability of $\omega \to \mathbf{1}_{\mathrm{sc}}(H_\omega)$ is usually based on Carmona's lemma: Let μ be a finite, positive Borel measure on $\mathbb{R}$, and let $\mathcal{I}$ be the set of finite unions of open intervals, each of which has rational endpoints. Then, for any Borel set B,

$$\mu_{\mathrm{sing}}(B) = \lim_{n\to\infty} \sup_{I\in\mathcal{I},|I|<1/n} \mu(B \cap I).$$

Prove Carmona's lemma and using this result show that $\omega \to \mathbf{1}_{\mathrm{sc}}(H_\omega)$ is measurable. Hint: See Section 9.1 in [CFKS].

[4] Recall that $M(\mathbb{R})$ is the Banach space of all complex measures on $\mathbb{R}$. Assume that $\omega \to \mu_\omega \in M(\mathbb{R})$ is measurable. Prove that this function is also measurable w.r.t. the uniform topology of $M(\mathbb{R})$.

[5] Assime that $\omega \to H_\omega$ is selfadjoint measurable. Prove that $\omega \to \mathbf{1}_{\mathrm{pp}}(H_\omega)$ is measurable by using Simon's local Wiener theorem (Theorem 5 and the homework problem 5 in Lecture 2).

[6] The proof of Proposition 1.3.6. is very compact. Please write down all the details.

6. Project.

Please start reading Sections 9 and 10 in [CFKS]. You should finish these two sections by the end of the course.