MATH 743
SPECTRAL THEORY OF ANDERSON TYPE HAMILTONIANS
LECTURE 3

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Preliminaries 3: Selfadjoint operators, spectral theory

O.1 References

[RS1] Reed M., Simon B.: *Methods of Modern Mathematical Physics, I. Functional Analysis*, Second edition, London, Academic Press (1980).

[RS2] Reed M., Simon B.: *Methods of Modern Mathematical Physics, II. Fourier Analysis, Self-Adjointness*, London, Academic Press (1975).

[RS3] Reed M., Simon B.: *Methods of Modern Mathematical Physics, III. Scattering Theory*, London, Academic Press (1978).

[RS4] Reed M., Simon B.: *Methods of Modern Mathematical Physics, IV. Analysis of Operators*, London, Academic Press (1978).

[Da] Davies E.B.: *Spectral Theory and Differential Operators*, Cambridge studies in advanced mathematics **42**, Cambridge University Press (1995).

[Kato] Kato, T.: *Perturbation Theory for Linear Operators*, second edition, Springer-Verlag, Berlin (1976).

[CL] Carmona R., Lacroix J.: *Spectral Theory of Random Schrödinger Operators*, Birkhäuser, Boston (1990).

1.1. Introduction

This lecture is a telegraphic introduction to the spectral theory of selfadjoint operators. Since all the students in the class have been already exposed to the spectral theory (and in particular to [RS1]), in this lecture I will just briefly recall the results we will need in the rest of the course.

Some proofs are given in detail, and in particular I have given a complete proof of the spectral theorem. The proof in the notes emphasizes the role of cyclic subspaces as the "basic units" of spectral theory. This point of view is important in study of the Anderson type Hamiltonians.

The Subsection 1.9 deals with the notion of the essential support of the absolutely continuous spectrum which is rarely discussed in books (see however [CL]). I have assumed that most of the students are not familiar with this notion. The essential support of the a.c. spectrum will play an important role in our study of the spectral structure of Anderson type Hamiltonians and it is very important that you master this notion. I have on purpose left for homework all the proofs in Subsection 1.9—they are very easy and instructive and it is important that you work them out.

The homework consists of two parts. First, you are asked to prove several results discussed in the Lecture. Secondly, I have included in the homework some more advanced topics (spectral theory of closed operators) which might be useful in your research (they will not be needed for the course).

At the end of the Lecture there are two projects. The first deals with scattering theory and the second with the Helffer-Sjöstrand formula.

1.2. Basic notions

Let $\mathcal{H}$ be a Hilbert space. We denote the inner product by $(\cdot|\cdot)$ (the inner product is linear w.r.t. the second variable).

A linear operator on $\mathcal{H}$ is a pair $(A, \mathrm{Dom}(A))$, where $\mathrm{Dom}(A) \subset \mathcal{H}$ is a vector subspace and $A : \mathrm{Dom}(A) \to \mathcal{H}$ is a linear map. We set

$$\mathrm{Ker}A = \{\psi \in \mathrm{Dom}(A) \ : \ A\psi = 0\}, \qquad \mathrm{Ran}A = \{A\psi \ : \ \psi \in \mathrm{Dom}(A)\}.$$

An operator A is densely defined if $\mathrm{Dom}(A)$ is dense in $\mathcal{H}$.

If A and B are linear operator, then $A + B$ is defined on $\mathrm{Dom}(A + B) := \mathrm{Dom}(A) \cap \mathrm{Dom}(B)$ in the obvious way. Similarly, $\mathrm{Dom}(AB) = \{\psi \ : \ \psi \in \mathrm{Dom}(B), B\psi \in \mathrm{Dom}(A)\}$, and $(AB)\psi = A(B\psi)$. $A = B$ if $\mathrm{Dom}(A) = \mathrm{Dom}(B)$ and $A\psi = B\psi$. The operator B is called an extension of A (one writes $A \subset B$) if $\mathrm{Dom}(A) \subset \mathrm{Dom}(B)$ and $A\psi = B\psi$ for $\psi \in \mathrm{Dom}(A)$.

The operator A is called bounded if $\mathrm{Dom}(A) = \mathcal{H}$ and

$$\|A\| := \sup_{\|\psi\|=1} \|A\psi\| < \infty. \tag{1}$$

We denote by $\mathcal{B}(\mathcal{H})$ the vector space of all bounded operators on $\mathcal{H}$. $\mathcal{B}(\mathcal{H})$ with the norm (1) is a Banach space. If A is densely defined and there is C such that for all $\psi \in \mathrm{Dom}(A)$, $\|A\psi\| \leq C\|\psi\|$, then A has a unique extension to a bounded operator on $\mathcal{H}$. An operator $P \in \mathcal{B}(\mathcal{H})$ is called a projection if $P^2 = P$. An operator $U \in \mathcal{B}(\mathcal{H})$ is called unitary if U is onto and $(U\phi|U\psi) = (\phi|\psi)$ for all $\phi, \psi \in \mathcal{H}$.

The graph of a linear operator A is defined by

$$\Gamma(A) = \{(\psi, A\psi) \ : \ \psi \in \mathrm{Dom}(A)\} \subset \mathcal{H} \oplus \mathcal{H}.$$

Note that $A \subset B$ if $\Gamma(A) \subset \Gamma(B)$. A linear operator A is called *closed* if $\Gamma(A)$ is a closed subset of $\mathcal{H} \oplus \mathcal{H}$. In other words, A is closed iff for any sequence $\phi_n \in \mathrm{Dom}(A)$ such that $\phi_n \to \phi$ and $A\phi_n \to \psi$, we have $\phi \in \mathrm{Dom}(A)$ and $A\phi = \psi$.

A is called closable if it has a closed extension. If A is closable, its smallest closed extension is called the closure of A and is denoted by $\overline{A}$. It is not difficult to show that A is closable iff $\overline{\Gamma(A)}$ is the graph of a linear operator and in this case $\Gamma(\overline{A}) = \overline{\Gamma(A)}$.

Let A be closed. A subset $D \subset \mathrm{Dom}(A)$ is called a *core* for A is $\overline{A \restriction D} = A$.

Let A be a densely defined linear operator. Its adjoint, A^*, is defined as follows: $\mathrm{Dom}(A^*)$ is the set of all $\phi \in \mathcal{H}$ for which there exists $\psi \in \mathcal{H}$ such that

$$(A\varphi|\phi) = (\varphi|\psi) \qquad \text{for all } \varphi \in \mathrm{Dom}(A).$$

Obviously, such ψ is unique and $\mathrm{Dom}(A^*)$ is a vector subspace. We set $A^*\phi = \psi$. It may happen that $\mathrm{Dom}(A^*) = \{0\}$. If $\mathrm{Dom}(A^*)$ is dense, then $A^{**} = (A^*)^*$, etc.

Theorem 1.1.1. Let A be densely defined linear operator. Then:
(1) A^* is closed.
(2) A is closable iff $\mathrm{Dom}(A^*)$ is dense, and in this case $\overline{A} = A^{**}$.
(3) If A is closable, then $\overline{A}^* = A^*$.

Let A be a closed densely defined operator. A complex number z belongs to the resolvent set, denoted $\rho(A)$, if

$$A - z : \mathrm{Dom}(A) \to \mathcal{H}$$

is a bijection. Then, by the closed graph theorem, $(A-z)^{-1} \in \mathcal{B}(\mathcal{H})$. The spectrum of A, $\mathrm{sp}(A)$, is defined by

$$\mathrm{sp}(A) := \mathbb{C} \setminus \rho(A).$$

A point $z \in \mathbb{C}$ is called an eigenvalue of A if there is $\psi \in \mathrm{Dom}(A)$, $\psi \neq 0$, such that $A\psi = z\psi$. The set of all eigenvalues is called the point spectrum of A and is denoted by $\mathrm{sp_p}(A)$. Obviously, $\mathrm{sp_p}(A) \subset \mathrm{sp}(A)$. It is possible that $\mathrm{sp}(A) = \mathrm{sp_p}(A) = \mathbb{C}$. It is also possible that $\mathrm{sp}(A) = \emptyset$. (For simple examples see [RS1], Example 5 in Chapter VIII).

Theorem 1.2.2. Assume that $\rho(A)$ is non-empty. Then $\rho(A)$ is an open subset of $\mathbb{C}$ and the map

$$\rho(A) \ni z \to (A-z)^{-1} \in \mathcal{B}(\mathcal{H}),$$

is (norm) analytic. Moreover, if $z_1, z_2 \in \rho(A)$, then

$$(A-z_1)^{-1} - (A-z_2)^{-1} = (z_1 - z_2)(A-z_1)^{-1}(A-z_2)^{-1}.$$

The last relation is called the resolvent identity.

`Appendix: The notions of analyticity`

Let $\Omega \subset \mathbb{C}$ be an open set and X a Banach space. A function $f : \Omega \to X$ is called norm analytic if for all $z \in \Omega$ the limit

$$\lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

exists in the norm of X. f is called weakly analytic if $x^* \circ f : \Omega \to \mathbb{C}$ is analytic for all $x^* \in X^*$. Obviously, if f is norm analytic, then f is weakly analytic. The converse also holds and we have:

Theorem 2.2 f is norm analytic iff f is weakly analytic.

For the proof, see [RS1].

The mathematical theory of the Banach space valued analytic functions parallels the classical theory of analytic function. For example, if γ is a closed path in Ω, then

$$\oint_\gamma f(z)\mathrm{d}z = 0. \tag{2}$$

(The integral is defined in the usual way by the norm convergent Riemann sums. Check all the details). To prove (2), note that for $x^* \in X^*$,

$$x^* \left(\oint_\gamma f(z)\mathrm{d}z \right) = \oint_\gamma x^*(f(z))\mathrm{d}z = 0.$$

Since X^* separates points in X, (2) holds. Then, using (2) and repeating the standard proof one obtains the Cauchy integral formula,

$$\frac{1}{2\pi \mathrm{i}} \oint_{|w-z|=r} \frac{f(w)}{w-z}\mathrm{d}w = f(z).$$

Starting with the Cauchy integral formula one proves in the usual way that for $w \in \Omega$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-w)^n, \tag{3}$$

where $a_n \in X$. The power series converges and the representation (3) holds in the largest open disk centered at w and contained in Ω. As a little project, work out the details.

Concerning the Lecture 2, many results of harmonic analysis extend to Hilbert space valued functions, see M. Rosenblum, J. Rovnyak.: *Hardy Classes and Operator Theory*, Dover 1997. These extensions have important applications.

1.3. Elementary properties of selfadjoint operators

Let A be a densely defined operator on a Hilbert space $\mathcal{H}$. A is called symmetric iff $\forall \phi, \psi \in \mathrm{Dom}(A)$,

$$(A\phi|\psi) = (\phi|A\psi).$$

In other words, A is symmetric if $A \subset A^*$. Obviously, any symmetric operator is closable.

A densely defined operator A is called *selfadjoint* if $A = A^*$. A is selfadjoint iff A is symmetric and $\mathrm{Dom}(A) = \mathrm{Dom}(A^*)$.

Theorem 1.3.1. Let A be a symmetric operator on $\mathcal{H}$. Then the following statements are equivalent:
(1) A is selfadjoint.
(2) A is closed and $\mathrm{Ker}(A^* \pm \mathrm{i}) = \{0\}$.
(3) $\mathrm{Ran}(A \pm \mathrm{i}) = \mathcal{H}$.

Remark. In (2) and (3) $\pm\mathrm{i}$ can be replaced by $z, \overline{z}$, for any $z \in \mathbb{C} \setminus \mathbb{R}$.

A symmetric operator A is called is essentially selfadjoint if $\overline{A}$ is selfadjoint.

Theorem 1.3.2. Let A be a symmetric operator on $\mathcal{H}$. Then the following statements are equivalent:
(1) A is essentially selfadjoint.
(2) $\mathrm{Ker}(A^* \pm \mathrm{i}) = \{0\}$.
(3) $\mathrm{Ran}(A \pm \mathrm{i})$ is dense in $\mathcal{H}$.

Theorem 1.3.3. Let A be selfadjoint. Then:
(1) If $z = x + \mathrm{i}y$, then for $\psi \in \mathrm{Dom}(A)$,

$$\|(A - z)\psi\|^2 = \|(A - x)\psi\|^2 + y^2\|\psi\|^2.$$

(2) $\mathrm{sp}(A) \subset \mathbb{R}$ and for $z \in \mathbb{C} \setminus \mathbb{R}$, $\|(A - z)^{-1}\| \leq |\mathrm{Im}z|^{-1}$.
(3) If $\lambda_1, \lambda_2 \in \mathrm{sp_p}(A)$, $\lambda_1 \neq \lambda_2$, and ψ_1, ψ_2 are corresponding eigenvectors, then $\psi_1 \perp \psi_2$.

Proof. (1) This is an explicit computation:

$$\begin{aligned}
\|(A - x - \mathrm{i}y)\psi\|^2 &= ((A - x - \mathrm{i}y)\psi|(A - x - \mathrm{i}y)\psi) \\
&= \|(A - x)\psi\|^2 + y^2\|\psi\|^2 + \mathrm{i}y((A - x)\psi|\psi) - \mathrm{i}y((A - x)\psi|\psi) \\
&= \|(A - x)\psi\|^2 + y^2\|\psi\|^2.
\end{aligned}$$

(2) Let $z \in \mathbb{C} \setminus \mathbb{R}$. By (1), if $(A - z)\psi = 0$, then $\psi = 0$, and so $A - z : \mathrm{Dom}(A) \to \mathcal{H}$ is one-one. $\mathrm{Ran}(A - z) = \mathcal{H}$ by Theorem 1.3.1. Let us prove this fact directly. Let us first show that $\mathrm{Ran}(A - z)$ is dense. Let $\psi \in \mathcal{H}$ such that $((A - z)\phi|\psi) = 0$ for all $\phi \in \mathrm{Dom}(A)$. Then $\psi \in \mathrm{Dom}(A)$ and $(\psi|A\psi) = \overline{z}\|\psi\|^2$. Since $(\psi|A\psi) \in \mathbb{R}$ and $\mathrm{Im}z \neq 0$, $\psi = 0$. Hence, $\mathrm{Ran}(A - z)$ is dense. Let $\psi_n = (A - z)\phi_n$ be a Cauchy sequence. Then, by (1), ϕ_n is also a Cauchy sequence, and since A is closed, $\mathrm{Ran}(A - z)$ is closed. Hence, $\mathrm{Ran}(A - z) = \mathcal{H}$ and $z \in \rho(A)$. Finally, the estimate $\|(A - z)^{-1}\| \leq |\mathrm{Im}z|^{-1}$ is an immediate consequence of (1).
(3) $\lambda_1(\psi_1|\psi_2) = (A\psi_1|\psi_2) = (\psi_1|A\psi_2) = \lambda_2(\psi_1|\psi_2)$. Since $\lambda_1 \neq \lambda_2$, $(\psi_1|\psi_2) = 0$. □

A selfadjoint operator A is called positive if $(\psi|A\psi) \geq 0$ for all $\psi \in \mathrm{Dom}(A)$. If A and B are bounded and selfadjoint, then obviously $A \pm B$ are also selfadjoint; we write $A \geq B$ if $A - B \geq 0$.

A selfadjoint projection P is called an orthogonal projection. In this case $\mathcal{H} = \mathrm{Ker}P \oplus \mathrm{Ran}P$. We write $\dim P = \dim \mathrm{Ran}P$.

1.4. Direct sums. Invariant subspaces.

Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and A_1, A_2 selfadjoint operators on $\mathcal{H}_1, \mathcal{H}_2$. Then, the operator $A = A_1 \oplus A_2$ with the domain $\mathrm{Dom}(A) = \mathrm{Dom}(A_1) \oplus \mathrm{Dom}(A_2)$ is selfadjoint. Obviously, $(A - z)^{-1} = (A_1 - z)^{-1} \oplus (A_2 - z)^{-1}$.

This elementary construction has a partial converse. Let A be a selfadjoint operator on a Hilbert space $\mathcal{H}$ and let $\mathcal{H}_1$ be a closed subspace of $\mathcal{H}$. The subspace $\mathcal{H}_1$ is *invariant* under A if for all $z \in \mathbb{C} \setminus \mathbb{R}$, $(A - z)^{-1}\mathcal{H}_1 \subset \mathcal{H}_1$. Obviously, if $\mathcal{H}_1$ is invariant under A, so is $\mathcal{H}_2 := \mathcal{H}_1^\perp$. Set $\mathrm{Dom}(A_i) = \mathrm{Dom}(A) \cap \mathcal{H}_i, A_i\psi = A\psi$, $i = 1, 2$. Then A_1, A_2 are selfadjoint operators on $\mathcal{H}_1$, $\mathcal{H}_2$ and $A = A_1 \oplus A_2$. We will call A_1 the restriction of A to the invariant subspace $\mathcal{H}_1$ and write $A_1 = A \restriction \mathcal{H}_1$.

Let Γ be a countable set and $\mathcal{H}_n$, $n \in \Gamma$, a collection of Hilbert spaces. The direct sum of this collection,

$$\mathcal{H} = \bigoplus_n \mathcal{H}_n,$$

is the set of all sequences $\{\psi_n\}_{n\in\Gamma}$ such that $\psi_n \in \mathcal{H}_n$ and

$$\sum_{n\in\Gamma} \|\psi_n\|^2_{\mathcal{H}_n} < \infty.$$

$\mathcal{H}$ is a Hilbert space with the inner product

$$(\phi|\psi) = \sum_{n\in\Gamma} (\phi_n|\psi_n)_{\mathcal{H}_n}.$$

Let $B_n \in \mathcal{B}(\mathcal{H}_n)$ and assume that $\sup_n \|B_n\| < \infty$. Then $B\{\psi_n\}_{n\in\Gamma} = \{B_n\psi_n\}_{n\in\Gamma}$ is a bounded operator on $\mathcal{H}$ and $\|B\| = \sup_n \|B_n\|$.

Proposition 1.4.1. Let A_n be selfadjoint operators on $\mathcal{H}_n$. Set

$$\mathrm{Dom}(A) = \{\psi = \{\psi_n\} \in \mathcal{H} \,:\, \psi_n \in \mathrm{Dom}(A_n), \sum_n \|A_n\psi_n\|^2_{\mathcal{H}_n} < \infty\},$$

$A\psi = \{A_n\psi_n\}$. Then A is a selfadjoint operator on $\mathcal{H}$. We write $A = \oplus_{n\in\Gamma} A_n$. Moreover:
(1) For $z \in \mathbb{C} \setminus \mathbb{R}$, $(A - z)^{-1} = \oplus_n (A_n - z)^{-1}$.
(2) $\mathrm{sp}(A) = \overline{\cup_n \mathrm{sp}(A_n)}$.

The proof of Proposition 1.4.1 is easy and is left for homework.

Proposition 1.4.1 has a converse which we will discuss in the next subsection.

1.5. Cyclic spaces. The decomposition theorem.

Let $\mathcal{H}$ be a separable Hilbert space and A a selfadjoint operator on $\mathcal{H}$. A collection of vectors $\mathcal{C} = \{\psi_n\}_{n\in\Gamma}$, where Γ is a countable set, is called *cyclic* for A if the closure of the linear span of the set of vectors

$$\{\psi_n, (A-z)^{-1}\psi_n \,:\, n \in \Gamma, z \in \mathbb{C}\setminus\mathbb{R}\}$$

is equal to $\mathcal{H}$. A cyclic set for A always exists (take an orthonormal basis for $\mathcal{H}$). If $\mathcal{C} = \{\psi\}$, then ψ is called a cyclic vector for A.

Theorem 1.5.1. (The decomposition theorem.) Let $\mathcal{H}$ be a separable Hilbert space and A a selfadjoint operator on $\mathcal{H}$. Then there exists a countable set Γ, a collection of mutually orthogonal closed subspaces $\{\mathcal{H}_n\}_{n\in\Gamma}$ of $\mathcal{H}$ ($\mathcal{H}_n \perp \mathcal{H}_m$ if $n \neq m$), and selfadjoint operators A_n on $\mathcal{H}_n$ such that:
(1) For all $n \in \Gamma$ there is $\psi_n \in \mathcal{H}_n$ cyclic for A_n.
(2) $\mathcal{H} = \oplus_n \mathcal{H}_n$ and $A = \oplus_n A_n$.

Proof. Let $\{\phi_n : n = 1, 2, \cdots\}$ be given cyclic set for A. Set $\psi_1 := \phi_1$ and let $\mathcal{H}_1$ be the cyclic space spanned by A and ψ_1, i.e., $\mathcal{H}_1$ is the closure of the linear span of the set of vectors $\{\psi_1, (A-z)^{-1}\psi_1 : z \in \mathbb{C}\setminus\mathbb{R}\}$. Obviously, $\mathcal{H}_1$ is invariant under A. Set $A_1 := A \restriction \mathcal{H}_1$.

We define ψ_n, $\mathcal{H}_n$ and A_n inductively as follows. If $\mathcal{H}_1 \neq \mathcal{H}$, let ϕ_{n_2} be the first element of the sequence $\{\phi_2, \phi_3, \cdots\}$ which is not in $\mathcal{H}_1$. Decompose $\phi_{n_2} = \phi'_{n_2} + \phi''_{n_2}$, where $\phi'_{n_2} \in \mathcal{H}_1$ and $\phi''_{n_2} \in \mathcal{H}_1^\perp$. Set $\psi_2 := \phi''_{n_2}$ and let $\mathcal{H}_2$ be the cyclic space spanned by A and ψ_2. It follows from the resolvent identity that $\mathcal{H}_1 \perp \mathcal{H}_2$. Set $A_2 := A \restriction \mathcal{H}_2$. In this way we inductively define $\psi_n, \mathcal{H}_n, A_n$, $n \in \Gamma$, where Γ is a finite set $\{1, \cdots, N\}$ or $\Gamma = \mathbb{N}$, and $\{\phi_n\}_{n\in\Gamma} \subset \cup_{n\in\Gamma}\mathcal{H}_n$. Hence, (1) holds and $\mathcal{H} = \oplus_n \mathcal{H}_n$.

To prove the second part of (2), note that by the construction

$$(A-z)^{-1} = \bigoplus_n (A_n - z)^{-1}.$$

If $\tilde{A} = \oplus A_n$, then by Proposition 1.4.1., $\tilde{A}$ is selfadjoint and $(\tilde{A}-z)^{-1} = \oplus_n (A_n - z)^{-1}$. Hence $A = \tilde{A}$.

1.6. The basic example. The spectral theorem.

Theorem 1.6.1. Let $(M, \mathcal{F})$ be a measure space and μ a finite positive measure on $(M, \mathcal{F})$. Let $f : M \to \mathbb{R}$ be a measurable function and let A_f be a linear operator on $L^2(M, \mathrm{d}\mu)$ defined by

$$\mathrm{Dom}(A_f) = \{\psi \in L^2(M, \mathrm{d}\mu) \,:\, f\psi \in L^2(M, \mathrm{d}\mu)\}, \qquad A_f\psi = f\psi.$$

Then:
(1) A_f is selfadjoint.
(2) A_f is bounded iff $f \in L^\infty(M, \mathrm{d}\mu)$, and in this case $\|A_f\| = \|f\|_\infty$.
(3) $\mathrm{sp}(A_f)$ is equal to the essential range of f:

$$\mathrm{sp}(A_f) = \{\lambda \in \mathbb{R} \, : \, \mu(f^{-1}(\lambda - \epsilon, \lambda + \epsilon)) > 0 \text{ for all } \epsilon > 0\}.$$

The proof of this theorem is left for homework.

The content of the spectral theorem for selfadjoint operators is that *any* selfadjoint operator is unitarily equivalent to A_f for some f.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. A linear bijection $U : \mathcal{H}_1 \to \mathcal{H}_2$ is called unitary if for all $\phi, \psi \in \mathcal{H}_1$, $(U\phi|U\psi)_{\mathcal{H}_2} = (\phi|\psi)_{\mathcal{H}_1}$. Let A_1, A_2 be linear operators on $\mathcal{H}_1, \mathcal{H}_2$. The operators A_1, A_2 are unitarily equivalent if there exists unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $U\mathrm{Dom}(A_1) = \mathrm{Dom}(A_2)$ and $UA_1U^{-1} = A_2$.

Theorem 1.6.2. (Spectral theorem for selfadjoint operators) Let A be a selfadjoint operator on a Hilbert space $\mathcal{H}$. Then there is a measure space $(M, \mathcal{F})$, a finite positive measure μ and measurable function $f : M \to \mathbb{R}$ such that A is unitarily equivalent to the operator A_f on $L^2(M, \mathrm{d}\mu)$.

We will prove spectral theorem only for separable Hilbert spaces.

In the literature one can find many different proofs of Theorem 1.6.2. The proof we will give in this notes is constructive and allows to explicitely identify M and f while the measure μ is directly related to $(A - z)^{-1}$.

1.7. Proof of the spectral theorem—the cyclic case.

Let A be a selfadjoint operator on a Hilbert space $\mathcal{H}$ and $\psi \in \mathcal{H}$.

Theorem 1.7.1. There exists a unique finite positive Borel measure μ_ψ on $\mathbb{R}$ such that

$$(\psi|(A - z)^{-1}\psi) = \int_{\mathbb{R}} \frac{\mathrm{d}\mu_\psi(t)}{t - z}, \qquad z \in \mathbb{C} \setminus \mathbb{R}. \tag{4}$$

The measure μ_ψ is called the spectral measure for A and ψ.

Proof. Set $U(z) := (\psi|(A - z)^{-1}\psi)$ and $V(z) = \mathrm{Im}U(z)$. It follows from the resolvent identity that

$$V(x + \mathrm{i}y) = y\|(A - x - \mathrm{i}y)^{-1}\psi\|^2,$$

and so V is harmonic and strictly positive in $\mathbb{C}_+$. By Theorem 1.7 in Lec 2, there exists a unique finite positive Borel measure μ_ψ on $\mathbb{R}$ such that for $y > 0$,

$$V(x + \mathrm{i}y) = P_{\mu_\psi}(x + \mathrm{i}y) = y \int_{\mathbb{R}} \frac{\mathrm{d}\mu_\psi(t)}{(x - t)^2 + y^2}.$$

Since $V(x - \mathrm{i}y) = -V(x + \mathrm{i}y)$, the same representation holds for $y < 0$. The functions

$$F_{\mu_\psi}(z) = \int_{\mathbb{R}} \frac{\mathrm{d}\mu_\psi(t)}{t - z}$$

and $U(z)$ are analytic in $\mathbb{C} \setminus \mathbb{R}$, have equal imaginary parts, and vanish as $|\mathrm{Im} z| \to \infty$. Hence, (4) holds. □

Corollary 1.7.2. Let $\varphi, \psi \in \mathcal{H}$. Then there exists a unique complex measure $\mu_{\varphi,\psi}$ on $\mathbb{R}$ such that

$$(\varphi|(A - z)^{-1}\psi) = \int_{\mathbb{R}} \frac{\mathrm{d}\mu_{\varphi,\psi}(t)}{t - z}, \qquad z \in \mathbb{C} \setminus \mathbb{R}. \tag{5}$$

Proof. The uniqueness is obvious (the set of functions $\{(x - z)^{-1} : z \in \mathbb{C} \setminus \mathbb{R}\}$ is dense in $C_0(\mathbb{R})$). The existence follows from the polarization identity:

$$\begin{aligned} 4(\varphi|(A - z)^{-1}\psi) &= (\varphi + \psi|(A - z)^{-1}(\varphi + \psi)) - ((\varphi - \psi)|(A - z)^{-1}(\varphi - \psi)) \\ &\quad + \mathrm{i}((\varphi - \mathrm{i}\psi)|(A - z)^{-1}(\varphi - \mathrm{i}\psi)) - \mathrm{i}(\varphi + \mathrm{i}\psi|(A - z)^{-1}(\varphi + \mathrm{i}\psi)). \end{aligned}$$

In particular,

$$\mu_{\varphi,\psi} = \frac{1}{4}\left(\mu_{\varphi+\psi} - \mu_{\varphi-\psi} + \mathrm{i}(\mu_{\varphi-\mathrm{i}\psi} - \mu_{\varphi+\mathrm{i}\psi})\right). \tag{6}$$

□

The main result of this subsection is:

Theorem 1.7.3. Assume that ψ is a cyclic vector for A. Then A is unitarily equivalent to the operator of multiplication by x on $L^2(\mathbb{R}, \mathrm{d}\mu_\psi)$. In particular, $\mathrm{sp}(A) = \mathrm{supp}\,\mu_\psi$.

Proof. Clearly, we may assume that $\psi \neq 0$. Note that $(A - z)^{-1}\psi = (A - w)^{-1}\psi$ iff $z = w$.

For $z \in \mathbb{C} \setminus \mathbb{R}$ we set $r_z(x) = (x - z)^{-1}$. $r_z \in L^2(\mathbb{R}, \mathrm{d}\mu_\psi)$ and the linear span of $\{r_z\}_{z \in \mathbb{C}\setminus\mathbb{R}}$ is dense in $L^2(\mathbb{R}, \mathrm{d}\mu_\psi)$. Set

$$U\psi = 1, \qquad U(A - z)^{-1}\psi = r_z.$$

Let us prove that

$$(r_z|r_w)_{L^2(\mathbb{R}, \mathrm{d}\mu_\psi)} = ((A - z)^{-1}\psi|(A - w)^{-1}\psi). \tag{7}$$

If $\overline{z} \neq w$, then

$$\begin{aligned}(r_z|r_w)_{L^2(\mathbb{R},\mathrm{d}\mu_\psi)} &= \int_{\mathbb{R}} r_{\overline{z}} r_w \mathrm{d}\mu = \frac{1}{\overline{z}-w}\int_{\mathbb{R}}(r_{\overline{z}} - r_w)\mathrm{d}\mu_\psi \\ &= \frac{1}{\overline{z}-w}\left[(\psi|(A-\overline{z})^{-1}\psi) - (\psi|(A-w)^{-1}\psi)\right] \\ &= ((A-z)^{-1}\psi|(A-w)^{-1}\psi).\end{aligned}$$

By a limiting argument, (7) also holds if $\overline{z} = w$. Hence, U extends to a unitary map $U : \mathcal{H} \to L^2(\mathbb{R}, \mathrm{d}\mu_\psi)$ and

$$U((A-z)^{-1}(A-w)^{-1}\psi) = r_z(x)r_w(x) = r_z(x)U((A-w)^{-1}\psi).$$

It follows that $(A-z)^{-1}$ is unitarily equivalent to the operator of multiplication by $(x-z)^{-1}$ on $L^2(\mathbb{R}, \mathrm{d}\mu_\psi)$. For any $\phi \in \mathcal{H}$,

$$U(A(A-z)^{-1}\phi) = U(\phi + z(A-z)^{-1}\phi) = (1 + z(x-z))U\phi = x(x-z)^{-1}U\phi,$$

and so A is unitarily equivalent to the operator of multiplication by x. □

Corollary 1.7.4. Let A be a selfadjoint operator on $\mathcal{H}$ and μ_ψ the spectral measure for A and ψ. Then $\mu_\psi(\mathbb{R}) = \|\psi\|^2$.

Proof. Let $y \neq 0$. The identity $A(A-\mathrm{i}y)^{-1}\psi = \psi + \mathrm{i}y(A-\mathrm{i}y)^{-1}\psi$ yields

$$\int_{\mathbb{R}} \frac{x}{x-\mathrm{i}y}\mathrm{d}\mu_\psi(x) = \|\psi\|^2 + \mathrm{i}y\int_{\mathbb{R}}\frac{\mathrm{d}\mu_\psi(x)}{x-\mathrm{i}y}.$$

Take $y \to \infty$ and use the dominated convergence theorem . □

1.8. Proof of the spectral theorem. The spectral subspaces.

Let A a selfadjoint operator on a separable Hilbert space $\mathcal{H}$. Let $\mathcal{H}_n$, A_n, ψ_n, $n \in \Gamma$ be as in the decomposition theorem (Theorem 1.5.1). Let $U_n : \mathcal{H}_n \to L^2(\mathbb{R}, \mathrm{d}\mu_{\psi_n})$ be unitary such that A_n is unitarily equivalent to the operator of multiplication by x. We denote this operator by $\tilde{A}_n$. Let $U = \oplus_n U_n$. Hence, an immediate consequence of the decomposition theorem and Theorem 1.7.3 is

Theorem 1.8.1. The map $U : \mathcal{H} \to \bigoplus_{n\in\Gamma} L^2(\mathbb{R}, \mathrm{d}\mu_{\psi_n})$ is unitary and A is unitarily equivalent to the operator $\bigoplus_{n\in\Gamma}\tilde{A}_n$. In particular,

$$\mathrm{sp}(A) = \overline{\bigcup_{n\in\Gamma} \operatorname{supp} \mu_{\psi_n}}.$$

This theorem is in fact the spectral theorem for selfadjoint operators. Note that if $\phi \in \mathcal{H}$ and $U\phi = \{\phi_n\}_{n\in\Gamma}$, then

$$\mu_\phi = \sum_{n\in\Gamma} \mu_{\phi_n}.$$

Theorem 1.6.2 is a reformulation of Theorem 1.8.1. Choose cyclic vectors ψ_n so that $\sum_{n\in\Gamma} \|\psi_n\|^2 < \infty$. For each $n \in \Gamma$, let $\mathbb{R}_n$ be a copy of $\mathbb{R}$ and

$$M := \bigcup_{n\in\Gamma} \mathbb{R}_n.$$

You may visualize M as follows: enumarate Γ so that $\Gamma = \{1, \cdots, N\}$ or $\Gamma = \mathbb{N}$ and set $\mathbb{R}_n = \{(n, x) : x \in \mathbb{R}\} \subset \mathbb{R}^2$. Hence, M is just a collection of lines in $\mathbb{R}^2$ parallel to the y-axis and going through the points $(n, 0), n \in \Gamma$. Let $\mathcal{F}$ be the collection of all sets $F \subset M$ such that $F \cap \mathbb{R}_n$ is Borel for all n. Then $\mathcal{F}$ is a σ-algebra and

$$\mu(F) := \sum_{n\in\Gamma} \mu_{\psi_n}(F \cap \mathbb{R}_n)$$

is a measure on M ($\mu(M) = \sum_{n\in\Gamma} \|\psi_n\|^2 < \infty$). Let $f : M \to \mathbb{R}$ be the identity function ($f(n, x) = x$). Then

$$L^2(M, \mathrm{d}\mu) = \bigoplus_{n\in\Gamma} L^2(\mathbb{R}, \mathrm{d}\mu_{\psi_n}), \qquad A_f = \bigoplus_{n\in\Gamma} \tilde{A}_n,$$

and Theorem 1.6.2 follows.

Set

$$\mu_{\mathrm{ac}}(F) := \sum_{n\in\Gamma} \mu_{\psi_n,\mathrm{ac}}(F \cap \mathbb{R}_n),$$

$$\mu_{\mathrm{sc}}(F) := \sum_{n\in\Gamma} \mu_{\psi_n,\mathrm{sc}}(F \cap \mathbb{R}_n),$$

$$\mu_{\mathrm{pp}}(F) := \sum_{n\in\Gamma} \mu_{\psi_n,\mathrm{pp}}(F \cap \mathbb{R}_n).$$

Then $L^2(M, \mathrm{d}\mu_{\mathrm{ac}})$, $L^2(M, \mathrm{d}\mu_{\mathrm{sc}})$, $L^2(M, \mathrm{d}\mu_{\mathrm{pp}})$ are closed subspace of $L^2(M, \mathrm{d}\mu)$ invariant under A_f and

$$L^2(M, \mathrm{d}\mu) = L^2(M, \mathrm{d}\mu_{\mathrm{ac}}) \oplus L^2(M, \mathrm{d}\mu_{\mathrm{sc}}) \oplus L^2(M, \mathrm{d}\mu_{\mathrm{pp}}).$$

Set

$$\mathcal{H}_{\mathrm{ac}} := U^{-1}L^2(M, \mathrm{d}\mu_{\mathrm{ac}}), \qquad \mathcal{H}_{\mathrm{sc}} := U^{-1}L^2(M, \mathrm{d}\mu_{\mathrm{sc}}), \qquad \mathcal{H}_{\mathrm{pp}} := U^{-1}L^2(M, \mathrm{d}\mu_{\mathrm{pp}}).$$

These subspaces are invariant under A; $\psi \in \mathcal{H}_{\rm ac}$ iff the spectral measure μ_ψ is absolutely continuous w.r.t. the Lebesgue measure, $\psi \in \mathcal{H}_{\rm sc}$ iff μ_ψ is singular continuous and $\psi \in \mathcal{H}_{\rm pp}$ iff μ_ψ is pure point (make sure that you understand this point). Obviously,

$$\mathcal{H} = \mathcal{H}_{\rm ac} \oplus \mathcal{H}_{\rm sc} \oplus \mathcal{H}_{\rm pp}.$$

The spectra

$$\mathrm{sp}_{\rm ac}(A) := \mathrm{sp}(A \restriction \mathcal{H}_{\rm ac}) = \overline{\bigcup_{n\in\Gamma} \mathrm{supp}\mu_{\psi_n,{\rm ac}}},$$

$$\mathrm{sp}_{\rm sc}(A) := \mathrm{sp}(A \restriction \mathcal{H}_{\rm sc}) = \overline{\bigcup_{n\in\Gamma} \mathrm{supp}\mu_{\psi_n,{\rm sc}}},$$

$$\mathrm{sp}_{\rm pp}(A) := \mathrm{sp}(A \restriction \mathcal{H}_{\rm pp}) = \overline{\bigcup_{n\in\Gamma} \mathrm{supp}\mu_{\psi_n,{\rm pp}}},$$

are called, respectively, the absolutely continuous, the singular continuous, and the pure point spectrum of A. Note that

$$\mathrm{sp}(A) = \mathrm{sp}_{\rm ac}(A) \cup \mathrm{sp}_{\rm sc}(A) \cup \mathrm{sp}_{\rm pp}(A),$$

and $\overline{\mathrm{sp}_{\rm p}(A)} = \mathrm{sp}_{\rm pp}(A)$. The singular and the continuous spectrum of A are defined by

$$\mathrm{sp}_{\rm sing}(A) := \mathrm{sp}_{\rm sc}(A) \cup \mathrm{sp}_{\rm pp}(A), \qquad \mathrm{sp}_{\rm cont}(A) := \mathrm{sp}_{\rm ac}(A) \cup \mathrm{sp}_{\rm sc}(A).$$

The subspaces $\mathcal{H}_{\rm ac}$, $\mathcal{H}_{\rm sc}$, $\mathcal{H}_{\rm pp}$ are called the spectral subspaces associated, respectively, to the absolutely continuous, singular continuous, and pure point spectrum. The projections on these subspaces are denoted by $\mathbf{1}_{\rm ac}(A)$, $\mathbf{1}_{\rm sc}(A)$, $\mathbf{1}_{\rm pp}(A)$. The spectral subspaces associated to the singular and the continuous spectrum are $\mathcal{H}_{\rm sing} = \mathcal{H}_{\rm sc} \oplus \mathcal{H}_{\rm pp}$ and $\mathcal{H}_{\rm cont} = \mathcal{H}_{\rm ac} \oplus \mathcal{H}_{\rm sc}$. The corresponding projections are $\mathbf{1}_{\rm sing}(A) := \mathbf{1}_{\rm sc}(A) + \mathbf{1}_{\rm pp}(A)$ and $\mathbf{1}_{\rm cont}(A) := \mathbf{1}_{\rm ac}(A) + \mathbf{1}_{\rm sc}(A)$.

When we wish to indicate the dependence of the spectral subspaces on the operator A, we will write $\mathcal{H}_{\rm ac}(A)$, etc.

1.9. Spectral measure for A. The essential support of the a.c. spectrum.

Let A be a selfadjoint operator on a separable Hilbert space $\mathcal{H}$ and let $\{\phi_n\}_{n\in\Gamma}$ be a cyclic set for A. Let $\{a_n\}_{n\in\Gamma}$ be a sequence such that $a_n > 0$ and

$$\sum_{n\in\Gamma} a_n \|\phi_n\|^2 < \infty.$$

The spectral measure for A, μ_A, is a Borel measure on $\mathbb{R}$ defined by

$$\mu_A(\cdot) = \sum_{n\in\Gamma} a_n \mu_{\phi_n}(\cdot).$$

Obviously, μ_A depends on the choice of $\{\phi_n\}$ and a_n. Two positive Borel measures ν_1 and ν_2 on $\mathbb{R}$ are called equivalent (we write $\nu_1 \sim \nu_2$) iff ν_1 and ν_2 have the same sets of measure zero.

Theorem 1.9.1. Let μ_A and ν_A be two spectral measures for A. Then $\mu_A \sim \nu_A$. Moreover, $\mu_{A,\mathrm{ac}} \sim \nu_{A,\mathrm{ac}}$, $\mu_{A,\mathrm{sc}} \sim \nu_{A,\mathrm{sc}}$, and $\mu_{A,\mathrm{pp}} \sim \nu_{A,\mathrm{pp}}$.

The proof is left for homework.

Theorem 1.9.2. Let μ_A be a spectral measure for A. Then

$$\mathrm{supp}\, \mu_{A,\mathrm{ac}} = \mathrm{sp}_{\mathrm{ac}}(A), \qquad \mathrm{supp}\, \mu_{A,\mathrm{sc}} = \mathrm{sp}_{\mathrm{sc}}(A), \qquad \mathrm{supp}\, \mu_{A,\mathrm{pp}} = \mathrm{sp}_{\mathrm{pp}}(A).$$

The proof is left for homework.

Let B_1 and B_2 be two Borel sets in $\mathbb{R}$. Set $B_1 \sim B_2$ iff the Lebesgue measure of the symmetric difference $(B_1 \setminus B_2) \cup (B_2 \setminus B_1)$ is zero. $\sim$ is the relation of the equivalence. Let μ_A be a spectral measure of A and $f(x)$ its Radon-Nikodym derivative w.r.t. the Lebesgue measure ($\mathrm{d}\mu_{A,\mathrm{ac}} = f(x)\mathrm{d}x$). The class of equivalence associated to $\{x : f(x) > 0\}$ is called the essential support of the absolutely continuous spectrum and is denoted by $\Sigma_{\mathrm{ac}}^{\mathrm{ess}}(A)$. With a slight abuse of notation we will also refer to a particular element of $\Sigma_{\mathrm{ac}}^{\mathrm{ess}}(A)$ as *an* essential support of the a.c. support (and denote it by the same symbol $\Sigma_{\mathrm{ac}}^{\mathrm{ess}}(A)$). For example, the set

$$\left\{ x : \lim_{r \downarrow 0} \frac{\mu_A(I(x,r))}{2r} \ \text{ exists, is finite and non-zero} \right\}$$

is an essential support of the absolutely continuous spectrum.

Note that the essential support of the a.c. spectrum is independent on the choice of μ_A.

Theorem 1.9.3. Let $\Sigma_{\mathrm{ac}}^{\mathrm{ess}}(A)$ be an essential support of the absolutely continuous spectrum. Then $\mathrm{cl}(\Sigma_{\mathrm{ac}}^{\mathrm{ess}}(A) \cap \mathrm{sp}_{\mathrm{ac}}(A)) = \mathrm{sp}_{\mathrm{ac}}(A)$.

The proof is left for homework.

Although the closure of an essential support $\Sigma_{\mathrm{ac}}^{\mathrm{ess}}(A) \subset \mathrm{sp}_{\mathrm{ac}}(A)$ equals $\mathrm{sp}_{\mathrm{ac}}(A)$, $\Sigma_{\mathrm{ac}}^{\mathrm{ess}}(A)$ could be substantially "smaller" then $\mathrm{sp}_{\mathrm{ac}}(A)$.

Exercise: Fix $0 < \epsilon < 1$. Construct an example of a selfadjoint operator A such that $\mathrm{sp}_{\mathrm{ac}}(A) = [0,1]$ and that the Lebesgue measure of $\Sigma_{\mathrm{ac}}^{\mathrm{ess}}(A)$ is equal to ϵ.

1.10. The functional calculus.

Let A be a selfadjoint operator on a separable Hilbert space $\mathcal{H}$. Let $U : \mathcal{H} \to L^2(M, \mathrm{d}\mu)$, f, and A_f be as in the spectral theorem. Let $C_{\mathrm{b}}(\mathbb{R})$ be the vector space of all bounded Borel functions on $\mathbb{R}$. For $h \in C_{\mathrm{b}}(\mathbb{R})$, consider the operator $A_{h\circ f}$. This operator is bounded and $\|A_{h\circ f}\| \leq \sup h(x)$. Set

$$h(A) = U^{-1} A_{h\circ f} U. \tag{8}$$

Let $\Phi : C_{\mathrm{b}}(\mathbb{R}) \to \mathcal{B}(\mathcal{H})$ be given by $\Phi(h) = h(A)$.

Theorem 1.10.1. (1) The map Φ is algebraic $*$-homomorphism.
(2) $\|\Phi(h)\| \leq \max |h(x)|$.
(3) $\Phi(r_z) = (A - z)^{-1}$ for all $z \in \mathbb{C} \setminus \mathbb{R}$.
(4) If $h_n(x) \to h(x)$ *for all* x, and $\sup_{n,x} |h_n(x)| < \infty$, then $h_n(A)\psi \to h(A)\psi$ for all ψ.
The map Φ is uniquely specified by (1)-(4). Moreover, it has the following additional properties:
(5) If $A\psi = \lambda\psi$, then $h(A)\psi = h(\lambda)\psi$.
(6) If $h \geq 0$, then $\Phi(h) \geq 0$.

Let $\mathcal{K} \subset \mathcal{H}$ be a closed subspace. If $\mathcal{K}$ is invariant under A, then for all $h \in C_{\mathrm{b}}(\mathbb{R})$, $h(A)\mathcal{K} \subset \mathcal{K}$.

For any Borel function $h : \mathbb{R} \to \mathbb{C}$ we define $h(A)$ by (8). Of course, $h(A)$ could be an unbounded operator. Note that $h_1(A)h_2(A) \subset h_1 \circ h_2(A)$, $h_1(A) + h_2(A) \subset (h_1 + h_2)(A)$. Also, $h(A)^* = \overline{h}(A)$ and $h(A)$ is self-adjoint iff $h(x) \in \mathbb{R}$ for μ-a.e. $x \in M$.

In fact, to define $h(A)$, we only need that $\mathrm{Ran} f \subset \mathrm{Dom} h$. Hence, if $A \geq 0$, we can define $\sqrt{A}$, if $A > 0$ we can define $\ln A$, etc.

The two classes of functions, characteristic functions and exponentials, play a particularly important role.

Let F be a Borel set in $\mathbb{R}$ and χ_F its characteristic function. The operator $\chi_F(A)$ is an orthogonal projection, called the spectral projection on the set F. In these notes we will use the notation $\mathbf{1}_F(A) \equiv \chi_F(A)$ and $\mathbf{1}_{\{e\}}(A) = \mathbf{1}_e(A)$. Note that $\mathbf{1}_e(A) \neq 0$ iff $e \in \mathrm{sp}_{\mathrm{p}}(A)$. By definition, the multiplicity of the eigenvalue e is $\dim \mathbf{1}_e(A)$

The subspace $\mathrm{Ran}\mathbf{1}_F(A)$ is invariant under A and

$$\mathrm{cl}(\mathrm{int}(F) \cap \mathrm{sp}(A)) \subset \mathrm{sp}(A \restriction \mathrm{Ran}\mathbf{1}_F(A)) \subset \mathrm{sp}(A) \cap \mathrm{cl}(F). \tag{9}$$

Note in particular that $e \in \mathrm{sp}(A)$ iff for all $\epsilon > 0$ $\mathrm{Ran}\mathbf{1}_{(e-\epsilon, e+\epsilon)}(A) \neq \{0\}$. The proof of (9) is left for homework.

Theorem 1.10.2. (Stone's formula) For $\psi \in \mathcal{H}$,

$$\lim_{y \downarrow 0} \frac{1}{2\pi \mathrm{i}} \int_a^b \left[(A - x - \mathrm{i}y)^{-1}\psi - (A - x + \mathrm{i}y)^{-1}\psi \right] \mathrm{d}x = \frac{1}{2} \left[\mathbf{1}_{[a,b]}(A)\psi + \mathbf{1}_{(a,b)}(A)\psi \right].$$

Proof. Since

$$\lim_{y \downarrow 0} \frac{y}{\pi} \int_a^b \frac{1}{(t-x)^2 + y^2} \mathrm{d}x = \begin{cases} 0 & \text{if } t \notin [a,b], \\ 1/2 & \text{if } t = a \text{ or } t = b, \\ 1 & \text{if } t \in (a,b), \end{cases}$$

the Stone formula follows from the spectral theorem. □

Another important class of functions are exponentials. For $t \in \mathbb{R}$, set $U(t) = \exp(-\mathrm{i}tA)$. Then $U(t)$ is a group of unitary operators on $\mathcal{H}$. The group $U(t)$ is strongly continuous: for all $\psi \in \mathcal{H}$,

$$\lim_{s \to t} U(s)\psi = U(t)\psi.$$

For $\psi \in \mathrm{Dom}(A)$, $t \to U(t)\psi$ is strongly differentiable,

$$\lim_{t \to 0} \frac{U(t)\psi - \psi}{t} = -\mathrm{i}A\psi. \tag{10}$$

Moreover, if the limit on the r.h.s. exists for some ψ, then $\psi \in \mathrm{Dom}(A)$ and (10) holds.

The above results have a converse:

Theorem 1.10.3. (Stone's theorem) Let $U(t)$ be a strongly continuous group on a separable Hilbert space $\mathcal{H}$. Then there is a self-adjoint operator A such that $U(t) = \exp(-\mathrm{i}tA)$.

1.11. The essential and discrete spectra. The Weyl criterion. The RAGE theorem.

Let A be a selfadjoint operator on a separable Hilbert space $\mathcal{H}$.

Theorem 1.11.1. (Weyl criterion 1.) $e \in \mathrm{sp}(A)$ iff there exists a sequence of unit vectors $\psi_n \in \mathrm{Dom}(A)$ such that

$$\lim_{n \to \infty} \|(A-e)\psi_n\| = 0. \tag{11}$$

Remark. A sequence of unit vectors for which (11) holds is called a Weyl sequence.

Proof. Recall that $e \in \mathrm{sp}(A)$ iff $\mathbf{1}_{(e-\epsilon, e+\epsilon)}(A) \neq 0$ for all $\epsilon > 0$.

Assume that $e \in \mathrm{sp}(A)$. Let $\psi_n \in \mathrm{Ran}\mathbf{1}_{(e-1/n, e+1/n)}(A)$ be unit vectors. Then, by the functional calculus,

$$\|(A-e)\psi_n\| \leq \sup_{x \in (e-1/n, e+1/n)} |x - e| \leq 1/n.$$

On the other hand, assume that there is a sequence ψ_n such that (11) holds and that $e \notin \mathrm{sp}(A)$. Then

$$\|\psi_n\| = \|(A-e)^{-1}(A-e)\psi_n\| \leq C\|(A-e)\psi_n\|,$$

and so $1 = \|\psi_n\| \to 0$, contradiction.□

The discrete spectrum of A, denoted $\mathrm{sp}_{\mathrm{disc}}(A)$, is the set of all isolated eigenvalues of finite multiplicity. Hence $e \in \mathrm{sp}_{\mathrm{disc}}(A)$ iff $1 \leq \dim \mathbf{1}_{(e-\epsilon,e+\epsilon)}(A) < \infty$ for all ϵ small enough. The essential spectrum of A is defined by

$$\mathrm{sp}_{\mathrm{ess}}(A) := \mathrm{sp}(A) \setminus \mathrm{sp}_{\mathrm{disc}}(A).$$

Hence, $e \in \mathrm{sp}_{\mathrm{ess}}(A)$ iff for all $\epsilon > 0$ $\dim \mathbf{1}_{(e-\epsilon,e+\epsilon)}(A) = \infty$. Obviously, $\mathrm{sp}_{\mathrm{ess}}(A)$ is a closed subset of $\mathbb{R}$.

Theorem 1.11.2. (Weyl criterion 2.) $e \in \mathrm{sp}_{\mathrm{ess}}(A)$ iff there exists an orthonormal sequence $\psi_n \in \mathrm{Dom}(A)$ ($\|\psi_n\| = 1$, $(\psi_n|\psi_m) = 0$ if $n \neq m$), such that

$$\lim_{n\to\infty} \|(A-e)\psi_n\| = 0. \tag{12}$$

Proof. Assume that $e \in \mathrm{sp}_{\mathrm{ess}}(A)$. Then $\dim \mathbf{1}_{(e-1/n,e+1/n)}(A) = \infty$ for all n, and we can choose an orthonormal sequence ψ_n such that $\psi_n \in \mathrm{Ran}\mathbf{1}_{(e-1/n,e+1/n)}(A)$. Clearly, $\|(A-e)\psi_n\| \leq 1/n$.

On the other hand, assume that there exists an orthonormal sequence ψ_n such that (12) holds and that $e \in \mathrm{sp}_{\mathrm{disc}}(A)$. Choose $\epsilon > 0$ such that that $\dim \mathbf{1}_{(e-\epsilon,e+\epsilon)}(A) < \infty$. Then, $\lim_{n\to\infty} \mathbf{1}_{(e-\epsilon,e+\epsilon)}(A)\psi_n = 0$ (explain why). Hence,

$$\lim_{n\to\infty} \|(A-e)\mathbf{1}_{\mathbb{R}\setminus(e-\epsilon,e+\epsilon)}(A)\psi_n\| = 0.$$

Since $(A-e) \restriction \mathrm{Ran}\mathbf{1}_{\mathbb{R}\setminus(e-\epsilon,e+\epsilon)}(A)$ is invertible and the norm of its inverse is $\leq 1/\epsilon$, we have that

$$\|\mathbf{1}_{\mathbb{R}\setminus(e-\epsilon,e+\epsilon)}(A)\psi_n\| \leq \epsilon^{-1}\|(A-e)\mathbf{1}_{\mathbb{R}\setminus(e-\epsilon,e+\epsilon)}\psi_n\|,$$

and so $\lim_{n\to\infty} \mathbf{1}_{\mathbb{R}\setminus(e-\epsilon,e+\epsilon)}(A)\psi_n = 0$. Hence $1 = \|\psi_n\| \to 0$, contradiction. □

RAGE Theorem 1.11.3. (1) Let K be a compact operator. Then for all $\psi \in \mathcal{H}_{\mathrm{cont}}$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \|K\mathrm{e}^{-\mathrm{i}tA}\psi\|^2 \mathrm{d}t = 0. \tag{13}$$

(2) The same result holds if K is bounded and $K(A+\mathrm{i})^{-1}$ is compact.

Remark. $\mathcal{H}_{\mathrm{cont}}$ can be characterized by (13): $\psi \in \mathcal{H}_{\mathrm{cont}}$ iff (13) holds for all compact K. Prove this fact.

Proof. (1) First, recall that any compact operator is a norm limit of finite rank operators. In other words, there exists vectors $\phi_n, \varphi_n \in \mathcal{H}$ such that $K_n = \sum_{j=1}^{n} (\phi_j|\cdot)\varphi_j$ converges in norm to K, $\lim_{n\to\infty} \|K - K_n\| = 0$. Hence, it suffices to prove the statement for the finite rank operators K_n. By induction and the triangle inequality, it suffices to prove the statement for the rank one operator $K = (\phi|\cdot)\varphi$. Thus, it suffices to show that for $\phi \in \mathcal{H}$ and $\psi \in \mathcal{H}_{\mathrm{cont}}$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |(\phi|\mathrm{e}^{-\mathrm{i}tA}\psi)|^2 \mathrm{d}t = 0.$$

Moreover, since

$$(\phi|\mathrm{e}^{-\mathrm{i}tA}\psi) = (\phi|\mathrm{e}^{-\mathrm{i}tA}\mathbf{1}_{\mathrm{cont}}(A)\psi) = (\mathbf{1}_{\mathrm{cont}}(A)\phi|\mathrm{e}^{-\mathrm{i}tA}\psi),$$

w.l.o.g we may assume that $\phi \in \mathcal{H}_{\mathrm{cont}}$. Finally, by the polarization identity (6), we may assume that $\varphi = \psi$. Since for $\psi \in \mathcal{H}_{\mathrm{cont}}$ the spectral measure μ_ψ has no atoms, the result follows from the Wiener theorem (Theorem 5 in Lec 1).

(2) Since $\mathrm{Dom}(A) \cap \mathcal{H}_{\mathrm{cont}}$ is dense in $\mathcal{H}_{\mathrm{cont}}$, it suffices to prove the statement for $\psi \in \mathrm{Dom}(A) \cap \mathcal{H}_{\mathrm{cont}}$. Write

$$\|K\mathrm{e}^{-\mathrm{i}tA}\psi\| = \|K(A+\mathrm{i})^{-1}\mathrm{e}^{-\mathrm{i}tA}(A+\mathrm{i})\psi\|,$$

and use (1). □

1.12. Stability

We will first discuss stability of selfadjointness—if A and B are selfadjoint operators, we wish to find conditions under which $A+B$ is selfadjoint (or essentially selfadjoint) on $\mathrm{Dom}(A) \cap \mathrm{Dom}(B)$. One obvious sufficient condition is that either A or B is bounded. A more refined result involves the notion of relative boundedness.

Let A and B be densely defined linear operators on a separable Hilbert space $\mathcal{H}$. The operator B is called *A-bounded* if $\mathrm{Dom}(A) \subset \mathrm{Dom}(B)$ and for some $a, b \in \mathbb{R}$ and all $\psi \in \mathrm{Dom}(A)$,

$$\|B\psi\| \le a\|A\psi\| + b\|\psi\|.$$

The infimum of such a's is called the relative bound of B w.r.t. A.

Theorem 1.12.1. (The Kato-Rellich theorem.) Suppose that A is selfadjoint, B is symmetric, and B is A-bounded with relative bound $a < 1$. Then $A + B$ is selfadjoint on $\mathrm{Dom}(A)$ and essentially selfadjoint on any core of A.

As a part of your homework, you are asked to read the proof of this theorem (Theorem X.12 in [RS2]). The Sections X.1–X.6 of [RS2] are devoted to study of selfadjointness.

We now turn to the stability of essential spectrum. The simplest result in this direction (and the only one we will need) is:

Theorem 1.12.2 (The Weyl theorem.) Let A and B be selfadjoint and B compact. Then $\mathrm{sp}_{\mathrm{ess}}(A) = \mathrm{sp}_{\mathrm{ess}}(A + B)$.

Proof. By symmetry, it suffices to prove that $\mathrm{sp}_{\mathrm{ess}}(A+B) \subset \mathrm{sp}_{\mathrm{ess}}(A)$. Let $e \in \mathrm{sp}_{\mathrm{ess}}(A+B)$ and let ψ_n be an orthonormal sequence such that

$$\lim_{n\to\infty} \|(A + B - e)\psi_n\| = 0.$$

Since ψ_n converges weakly to zero and B is compact, $B\psi_n \to 0$. Hence, $\|(A - e)\psi_n\| \to 0$ and $e \in \mathrm{sp}_{\mathrm{ess}}(A)$.$\Box$

The Section XIII.4 of [RS4] deals with various extensions and generalizations of Theorem 1.12.2.

We now discuss the stability of the absolutely continuous spectrum. A bounded selfadjoint operator $A \geq 0$ is *trace class* if

$$\mathrm{Tr}(A) := \sum_n (\psi_n|A\psi_n) < \infty,$$

where $\{\psi_n\}$ is an orthonormal basis of $\mathcal{H}$. The number $\mathrm{Tr}(A)$ does not depend on the choice of $\{\psi_n\}$. More generaly, a bounded selfadjoint operator A is trace class if $\mathrm{Tr}(|A|) < \infty$. A trace class operator is compact.

Theorem 1.12.3. (The Kato-Rosenblum theorem.) Let A and B be selfadjoint and B trace class. Then:
(1) $\mathrm{sp}_{\mathrm{ac}}(A + B) = \mathrm{sp}_{\mathrm{ac}}(A)$.
(2) The operators $A + B \restriction \mathcal{H}_{\mathrm{ac}}(A + B)$ and $A \restriction \mathcal{H}_{\mathrm{ac}}(A)$ are unitarily equivalent.
(3) $\Sigma_{\mathrm{ac}}^{\mathrm{ess}}(A + B) = \Sigma_{\mathrm{ac}}^{\mathrm{ess}}(A)$.

The Kato-Rosenblum theorem belongs to the part of mathematical physics called *scattering theory*. The entire volume [RS3] is devoted to the scattering theory. Your first project deals with scattering theory and the proof of Theorem 1.12.3.

In this course we will use Theorem 1.12.3 only in the case where B is a finite rank operator ($\dim \mathrm{Ran} B < \infty$). In this special case, the theorem has a simple proof based on the rank one perturbation theory. This proof will be discussed latter in the course.

Theorem 1.12.3 is an optimal result: see Sections X.2.1–X.2.2 in [Kato].

The singular and the pure point spectrum are in general unstable under perturbations—they may appear or dissapear under the influence of a rank one perturbation. We will discuss latter in the course criteria for "generic" stability of the singular and the pure point spectrum.

1.13. Homework

[0] Please read again Sections VII and VIII of [RS1]. Do all the exercises in these sections.

[1] Prove Proposition 1.4.1.

[2] Prove Theorem 1.6.1.

[3] Prove Theorem 1.9.1.

[4] Prove Theorem 1.9.2.

[5] Prove Theorem 1.9.3.

[6] Prove that $A \geq 0$ iff $\mathrm{sp}(A) \subset [0, \infty)$.

[7] Prove Relation (9).

[8] Read the proof of the Kato-Rellich theorem.

[9] Let A and B be selfadjoint operators such that $A \restriction \mathcal{H}_{\mathrm{ac}}(A)$ and $B \restriction \mathcal{H}_{\mathrm{ac}}(B)$ are unitarily equivalent. Prove that $\Sigma_{\mathrm{ac}}^{\mathrm{ess}}(A) = \Sigma_{\mathrm{ac}}^{\mathrm{ess}}(B)$.

The next set of problems deals with spectral theory of a closed operator A on a separable Hilbert space $\mathcal{H}$.

[10] Let $F \subset \mathrm{sp}(A)$ be an isolated, bounded subset of $\mathrm{sp}(A)$. Let γ be a closed simple path in the complex plane that separates F from $\mathrm{sp}(A) \setminus F$. Set

$$\mathbf{1}_F(A) = \frac{1}{2\pi \mathrm{i}} \oint_\gamma (z - A)^{-1} \mathrm{d}z.$$

(a) Prove that $\mathbf{1}_F(A)$ is a (not necessarily orthogonal) projection.
(b) Prove that $\mathrm{Ran}\mathbf{1}_F(A)$ and $\mathrm{Ker}\mathbf{1}_F(A)$ are complementary (not necessarily orthogonal) subspaces: $\mathrm{Ran}\mathbf{1}_F(A) + \mathrm{Ker}\mathbf{1}_F(A) = \mathcal{H}$ and $\mathrm{Ran}\mathbf{1}_F(A) \cap \mathrm{Ker}\mathbf{1}_F(A) = \{0\}$.

(c) Prove that $\mathrm{Ran}\mathbf{1}_F(A) \subset \mathrm{Dom}(A)$ and that $A : \mathrm{Ran}\mathbf{1}_F(A) \to \mathrm{Ran}\mathbf{1}_F(A)$. Prove that $A \restriction \mathrm{Ran}\mathbf{1}_F(A)$ is a bounded operator and that its spectrum is F.
(d) Prove that $\mathrm{Ker}\mathbf{1}_F(A) \cap \mathrm{Dom}(A)$ is dense and that

$$A \restriction \mathrm{Ker}\mathbf{1}_F(A) \cap \mathrm{Dom}(A) \to \mathrm{Ker}\mathbf{1}_F(A). \tag{14}$$

Prove that the operator (14) is closed and that its spectrum is $\sigma(A) \setminus F$.
(f) Assume that $F = \{z_0\}$ and that $\mathrm{Ran}\mathbf{1}_{z_0}(A)$ is finite dimensional. Prove that if $\psi \in \mathrm{Ran}\mathbf{1}_{z_0}(A)$, then $(A - z_0)^n\psi = 0$ for some n.
Hint: Consult Theorem XII.5 in [RS4].
Remark. The set of $z_0 \in \mathrm{sp}(A)$ which satisfy (f) is called the discrete spectrum of the closed operator operator A and is denoted $\mathrm{sp}_{\mathrm{disc}}(A)$. The essential spectrum is defined by $\mathrm{sp}_{\mathrm{ess}}(A) = \mathrm{sp}(A) \setminus \mathrm{sp}_{\mathrm{disc}}(A)$

[11] Prove that $\mathrm{sp}_{\mathrm{ess}}(A)$ is a closed subset of $\mathbb{C}$.

[12] Prove that $z \to (A-z)^{-1}$ is a meromorphic function on $\mathbb{C}\setminus \mathrm{sp}_{\mathrm{ess}}(A)$ with singularities at points $z_0 \in \mathrm{sp}_{\mathrm{disc}}(A)$. Prove that the negative coefficents of of the Laurent expansion at $z_0 \in \mathrm{sp}_{\mathrm{disc}}(A)$ are finite rank operators.
Hint: Consult Lemma 1 in [RS4], Section XIII.4.

[13] The numerical range of A is defined by $N(A) = \{(\psi|A\psi) : \psi \in \mathrm{Dom}(A)\}$. In general, $N(A)$ is neither open nor closed subset of $\mathbb{C}$. It is a deep result of Hausdorff that $N(A)$ is convex. Prove that if $\mathrm{Dom}(A) = \mathrm{Dom}(A^*)$, then $\mathrm{sp}(A) \subset \mathrm{cl}(N(A))$. This result is technically often useful. If you wish to learn more about numerical range, you may consult K. Gustafson, D.K.M. Rao.: Numerical Range. Universitext, Springer-Verlag (1996).

[14] If $z \in \mathrm{sp}(A)$, then in general there might not be any Weyl sequence ψ_n such that $\|(A-z)\psi_n\| \to 0$. Prove however that a Weyl sequence exists for every z on the topological boundary of $\mathrm{sp}(A)$. This is a bit difficult. Try to find a proof by reading the Section XIII.4 of [RS4].

1.14. Project 1 This projects deals with some elementary aspects of scattering theory. Let A and B be selfadjoint operators on a Hilbert space $\mathcal{H}$. Assume that for all $\psi \in \mathrm{Ran}\mathbf{1}_{\mathrm{ac}}(A)$ the limits

$$\Omega^{\pm}(A,B)\psi := \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tA}\mathrm{e}^{-\mathrm{i}tB}\psi \tag{15}$$

exists. The operators $\Omega^{\pm}(A,B) : \mathrm{Ran}\mathbf{1}_{\mathrm{ac}}(A) \to \mathcal{H}$ are called *wave operators.*
(1) Prove that

$$(\Omega^{\pm}(A,B)\phi|\Omega^{\pm}(A,B)\psi) = (\phi|\psi)$$

(obvious), that for any $f \in C_{\rm b}(\mathbb{R})$ $\Omega^{\pm}(A,B)f(A) = f(B)\Omega^{\pm}(A,B)$, and that

$$\mathrm{Ran}\Omega^{\pm}(A,B) \subset \mathbf{1}_{\rm ac}(B).$$

(2) The wave operators $\Omega^{\pm}(A,B)$ are called complete if $\mathrm{Ran}\Omega^{\pm}(A,B) = \mathrm{Ran}\mathbf{1}_{\rm ac}(B)$. Prove that $\Omega^{\pm}(A,B)$ are complete iff $\Omega^{\pm}(B,A)$ exists.
(3) Read enough about scattering so that you can understand the proof of the Kato-Rosenblum theorem: If $A - B$ is trace clace, then $\Omega^{\pm}(A,B)$ exist and are complete. The convenient references are [RS3] and [Kato], Chapter X.
(4) Kato's smooth perturbation theory: Read the Section XIII.7 in [RS4] until Theorem XIII.27. Read the proofs of the Theorems XIII.30 and XIII.31.

1.15. Project 2

Read the Chapter 2 of the Davies book [Da]. In this chapter Davies proves the spectral theorem using almost analytic extensions of smooth functions and the Helffer-Sjöstrand formula. This approach to the spectral theorem is conceptually interesing and technically powerful.