# MATH 743
# SPECTRAL THEORY OF ANDERSON TYPE HAMILTONIANS
# LECTURE 2

## Preliminaries 2: Harmonic analysis

### O.1 References

[Ru] Rudin W.: *Real and Complex Analysis*, 3rd edition, McGraw-Hill, 1987.

[Ka] Katznelson A.: *An Introduction to Harmonic Analysis*, Dover, New York, 1976.

[Ko] Koosis P.: *Introduction to $H_p$ Spaces*, Cambridge University Press, New York, Second edition 1998.

[RR] Rosenblum M., Rovnyak J.: *Topics in Hardy Classes and Univalent Functions*, Birkhäuser Advanced texts, Birkhäuser 1994.

[Zy] Zygmund A.: *Trigonometric series*, Second edition, Volumes I and II combined, Cambridge University Press 1993.

[Si1] Simon B.: $L^p$ norms of the Borel transforms and the decomposition of measures, Proc. Amer. Math. Soc. 123 (1995), 3749-3755, available at http://www.math.caltech.edu/people/biblio.html

[Si2] Simon B.: Spectral analysis of rank one perturbations and applications, CRM Lecture Notes Vol. 8, pp. 109-149, AMS, Providence, RI, 1995.

[Po] Poltoratskii A.G.: The boundary behavior of pseudocontinuable functions, St. Petersburgh Math. J. 5 (1994), 389-406, available at http://www.math.tamu.edu/ alexeip/publications.htm

[JL] Jaksic V., Last Y.: A new proof of Poltoratskii's theorem, to appear in J. Func. Anal., available on the course web site.

### 1.1. Introduction

In this section we will deal only with Borel measures on $\mathbb{R}$. We will use the shorthand $\mathbb{C}_+ = \{z : \mathrm{Im} z > 0\}$. We denote the Lebesgue measure by $m$ and write $\mathrm{d}m = \mathrm{d}x$.

Let $\mu$ be a complex measure or positive measure such that

$$\int_{\mathbb{R}} \frac{\mathrm{d}|\mu|(x)}{1+|x|}\mathrm{d}x < \infty. \tag{1}$$

The Borel transform of $\mu$ is defined by

$$F_\mu(z) = \int_{\mathbb{R}} \frac{\mathrm{d}\mu(x)}{x-z}, \qquad z \in \mathbb{C}_+. \tag{2}$$

The function $F_\mu(z)$ is analytic in $\mathbb{C}_+$. In the sequel, whenever we speak about the Borel transform of a positive measure we will assume that (1) is satisfied.

Let $\mu$ be a complex measure or positive measure such that

$$\int_{\mathbb{R}} \frac{\mathrm{d}|\mu|(x)}{1+x^2} < \infty. \tag{3}$$

The Poisson transform of $\mu$ is defined by

$$P_\mu(x+\mathrm{i}y) = y\int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(x-t)^2+y^2}, \qquad y>0.$$

The function $P_\mu(z)$ is harmonic in $\mathbb{C}_+$. If $\mu$ is the Lebesgue measure, then $P_\mu(z)=\pi$. If $\mu$ is a positive or signed measure, then $\mathrm{Im}F_\mu = P_\mu$. Note also that $F_\mu$ and $P_\mu$ are linear functions of $\mu$, i.e. for $\lambda_1, \lambda_2 \in \mathbb{C}$, $F_{\lambda_1\mu_1+\lambda_2\mu_2} = \lambda_1 F_{\mu_1} + \lambda_2 F_{\mu_2}$, $P_{\lambda_1\mu_1+\lambda_2\mu_2} = \lambda_1 P_{\mu_1} + \lambda_2 P_{\mu_2}$,

Our goal in this section is to study the boundary values of $P_\mu(x+\mathrm{i}y)$ and $F_\mu(x+\mathrm{i}y)$ as $y \downarrow 0$. More precisely, we wish to study how these boundary values reflect the properties of the measure $\mu$.

Although we will restrict ourselves to the radial limits, all the results discussed in this section hold for the non-tangential limits. The non-tangential limits play an important role in harmonic analysis. However, they are not needed for our applications and, for brevity and notational simplicity, we will restrict ourselves to the radial limits (all the proofs with minor changes extend to the non-tangential limits—see the homework).

**1.2. Poisson transforms and Radon-Nikodym derivatives**

This subsection is based on [JL]. Recall that $I(x,r) := (x-r, x+r)$.

**Lemma 1.** Let $\mu$ be a positive measure. Then for all $x \in \mathbb{R}$ and $y>0$,

$$\frac{1}{y}P_\mu(x+\mathrm{i}y) = \int_0^{1/y^2} \mu(I(x,\sqrt{u^{-1}-y^2}))\mathrm{d}u.$$

**Proof.** Note that

$$\begin{aligned}\int_0^{1/y^2} \mu(I(x,\sqrt{u^{-1}-y^2}))\mathrm{d}u &= \int_0^{1/y^2}\left[\int_{\mathbb{R}} \chi_{I(x,\sqrt{u^{-1}-y^2})}(t)\mathrm{d}\mu(t)\right]\mathrm{d}u \\ &= \int_{\mathbb{R}}\left[\int_0^{1/y^2} \chi_{I(x,\sqrt{u^{-1}-y^2})}(t)\mathrm{d}u\right]\mathrm{d}\mu(t).\end{aligned} \tag{4}$$

Since

$$|x-t| < \sqrt{u^{-1}-y^2} \qquad \iff \qquad 0 \le u < ((x-t)^2+y^2)^{-1},$$

we have

$$\chi_{I(x,\sqrt{u^{-1}-y^2})}(t) = \chi_{[0,((x-t)^2+y^2))^{-1})}(u),$$

and

$$\int_0^{1/y^2} \chi_{I(x,\sqrt{u^{-1}-y^2})}(t)\mathrm{d}u = ((x-t)^2+y^2))^{-1}.$$

Hence, the result follows from (4). $\square$

**Lemma 2.** Let $\nu$ be a complex and $\mu$ a positive measure. Then for all $x \in \mathbb{R}$ and $y > 0$,

$$\frac{|P_\nu(x+\mathrm{i}y)|}{P_\mu(x+\mathrm{i}y)} \le M_{\nu,\mu}(x).$$

**Proof.** Since $|P_\nu| \le P_{|\nu|}$, w.l.o.g we may assume that $\nu$ is positive. Also, we may assume that $x \in \operatorname{supp}\mu$ (otherwise $M_{\nu,\mu}(x) = \infty$ and there is nothing to prove). Since

$$\begin{aligned}\int_0^{1/y^2} \nu(I(x,\sqrt{u^{-1}-y^2}))\mathrm{d}u &= \int_0^{1/y^2} \frac{\nu(I(x,\sqrt{u^{-1}-y^2}))}{\mu(I(x,\sqrt{u^{-1}-y^2}))}\mu(I(x,\sqrt{u^{-1}-y^2}))\mathrm{d}u \\ &\le M_{\nu,\mu}(x)\int_0^{1/y^2} \mu(I(x,\sqrt{u^{-1}-y^2}))\mathrm{d}u,\end{aligned}$$

the result follows from Lemma 1.$\square$

**Lemma 3.** Let $\mu$ be a positive measure. Then for $\mu$-a.e. $x$,

$$\int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(x-t)^2} = \infty \tag{5}$$

The proof of this (easy) lemma is left for homework.

**Lemma 4.** Let $\mu$ be a positive measure and $f \in C_0(\mathbb{R})$. Then

$$\lim_{y\downarrow 0} \frac{P_{f\mu}(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)} = f(x) \qquad \text{for } \mu-\ a.e.\ x. \tag{6}$$

**Remark.** The relation (6) holds for all $x$ for which (5) holds. For example, if $\mu$ is the Lebesgue measure, then (6) holds for all $x$.
**Proof.** Note that

$$\left|\frac{P_{f\mu}(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)} - f(x)\right| \le \frac{P_{|f-f(x)|\mu}(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)}.$$

Fix $\epsilon > 0$ and let $\delta > 0$ be such that $|x-t| < \delta \Rightarrow |f(x) - f(t)| < \epsilon$. Let $M = \sup|f(t)|$ and

$$C = 2M \int_{|x-t|\geq\delta} \frac{\mathrm{d}\mu(t)}{(x-t)^2}.$$

Then

$$P_{|f-f(x)|\mu}(x+\mathrm{i}y) \leq \epsilon P_\mu(x+\mathrm{i}y) + Cy,$$

and

$$\left| \frac{P_{f\mu}(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)} - f(x) \right| \leq \epsilon + \frac{Cy}{P_\mu(x+iy)}.$$

Let $x$ be such that (5) holds. The monotone convergence theorem yields that

$$\lim_{y\downarrow 0} \frac{y}{P_\mu(x+iy)} = \left( \int \frac{\mathrm{d}\mu(t)}{(x-t)^2} \right)^{-1} = 0$$

and so for all $\epsilon > 0$,

$$\limsup_{y\downarrow 0} \left| \frac{P_{f\mu}(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)} - f(x) \right| \leq \epsilon.$$

This yields the statement.□

The main result of this subsection is:

**Theorem 1.** Let $\nu$ be a complex measure and $\mu$ a positive measure. Let $\nu = f\mu + \nu_{\mathrm{s}}$ be the Radon-Nikodym decomposition. Then:
(1)

$$\lim_{y\downarrow 0} \frac{P_\nu(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)} = f(x), \qquad \text{for } \mu - a.e.\ x.$$

In particular, $\nu \perp \mu$ iff

$$\lim_{y\downarrow 0} \frac{P_\nu(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)} = 0, \qquad \text{for } \mu - a.e.\ x.$$

(2) Assume in addition that $\nu$ is positive. Then

$$\lim_{y\downarrow 0} \frac{P_\nu(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)} = \infty, \qquad \text{for } \nu_{\mathrm{s}} - a.e.\ x.$$

**Proof.** The proof is very similar to the proof of Theorem 8 in Lecture 1.

(1) We will split the proof into two steps.

Step 1. Assume that $\nu \ll \mu$, namely that $\nu = f\mu$. Let $g_n$ be a continuous function with compact support such that $\int_{\mathbb{R}} |f - g_n| \mathrm{d}\mu < 1/n$. Set $h_n = f - g_n$. Then,

$$\left| \frac{P_{f\mu}(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)} - f(x) \right| \leq \frac{P_{|h_n|\mu}(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)} + \left| \frac{P_{g_n\mu}(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)} - g_n(x) \right| + |g_n(x) - f(x)|.$$

It follows from Lemmas 2 and 4 that for $\mu$-a.e. $x$,

$$\limsup_{y\downarrow 0} \left| \frac{P_{f\mu}(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y))} - f(x) \right| \leq M_{|h_n|\mu,\mu}(x) + |g_n(x) - f(x)|.$$

As in the proof of Theorem 8, there is a subsequence $n_j \to \infty$ such that $g_{n_j}(x) \to f(x)$ and $M_{|h_n|\mu,\mu}(x) \to 0$ for $\mu$-a.e. $x$, and (1) holds if $\nu \ll \mu$.

Step 2. To finish the proof of (1), it suffices to show that if $\nu$ is a finite positive measure such that $\nu \perp \mu$, then

$$\lim_{y\downarrow 0} \frac{P_\nu(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)} = 0, \qquad \text{for } \mu - \text{ a.e. } x. \tag{7}$$

Let $S$ be a Borel set such that $\mu(S) = 0$ and $\nu(\mathbb{R} \setminus S) = 0$. Then

$$\frac{P_\nu(x+\mathrm{i}y)}{P_\nu(x+\mathrm{i}y) + P_\mu(x+\mathrm{i}y)} = \frac{P_{\chi_S(\nu+\mu)}(x+\mathrm{i}y)}{P_{\nu+\mu}(x+\mathrm{i}y)}. \tag{8}$$

By Step 1,

$$\lim_{y\downarrow 0} \frac{P_{\chi_S(\nu+\mu)}(x+\mathrm{i}y)}{P_{\nu+\mu}(x+\mathrm{i}y)} = \chi_S(x), \qquad \text{for } \nu + \mu - \text{ a.e. } x. \tag{9}$$

Since $\chi_S(x) = 0$ for $\mu$-a.e. $x$,

$$\lim_{y\downarrow 0} \frac{P_\nu(x+\mathrm{i}y)}{P_\nu(x+\mathrm{i}y) + P_\mu(x+\mathrm{i}y)} = 0, \qquad \text{for } \mu - \text{ a.e. } x,$$

and (7) follows.

(2) Since $\nu$ is positive, $\nu(I(x,r)) \geq \nu_{\mathrm{s}}(I(x,r))$, and we may assume that $\nu \perp \mu$. By (8) and (9),

$$\lim_{y\downarrow 0} \frac{P_\nu(x+\mathrm{i}y)}{P_\nu(x+\mathrm{i}y) + P_\mu(x+\mathrm{i}y)} = 1, \qquad \text{for } \nu - \text{ a.e. } x,$$

and so

$$\lim_{y\downarrow 0} \frac{P_\mu(x+iy)}{P_\nu(x+\mathrm{i}y)} = 0, \qquad \text{for } \nu - \text{ a.e. } x.$$

This yields part (2). $\Box$

**1.3. Local $L^p$ norms, $0 < p < 1$.**

In this subsection we prove Theorem 3.1 of [Si1]. $\nu$ is a complex measure, $\mu$ is a positive measure and $\nu = f\mu + \nu_{\rm s}$ is the Radon-Nikodym decomposition.

**Theorem 2.** Let $A$ be a bounded Borel set and $0 < p < 1$. Then

$$\lim_{y\downarrow 0} \int_A \left| \frac{P_\nu(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)} \right|^p \mathrm{d}\mu(x) = \int_A |f(x)|^p \mathrm{d}\mu(x).$$

(Both sides are allowed to be $\infty$). In particular, $\nu \restriction A \perp \mu \restriction A$ iff for some $p \in (0,1)$,

$$\lim_{y\downarrow 0} \int_A \left| \frac{P_\nu(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)} \right|^p \mathrm{d}\mu(x) = 0.$$

**Proof.** By Theorem 1,

$$\lim_{y\downarrow 0} \left| \frac{P_\nu(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)} \right|^p = |f(x)|^p \qquad \text{for } \mu - \ a.e.\, x.$$

By Lemma 2,

$$\left| \frac{P_\nu(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)} \right|^p \leq |M_{\nu,\mu}(x)|^p.$$

Proposition 1 in the Lecture 1 and the dominated convergence theorem yield the statement. $\Box$

**1.4. Weak convergence**

Let $\nu$ be a complex or positive measure and

$$\mathrm{d}\nu_y(x) = \frac{1}{\pi} P_\nu(x+\mathrm{i}y)\mathrm{d}x. \tag{10}$$

**Theorem 3.** $\nu_y \to \nu$ weakly as $y \downarrow 0$. In other words, for any $f \in C_c(\mathbb{R})$ (continuous functions of compact support),

$$\lim_{y\downarrow 0} \int_{\mathbb{R}} f(x)\mathrm{d}\nu_y(x) = \int_{\mathbb{R}} f(x)\mathrm{d}\nu(x). \tag{11}$$

**Remark 1.** Let $\nu_{\rm sing}$ be the singular part of $\nu$ w.r.t. the Lebesgue measure. The limit and the integral on the r.h.s of (11) can be interchanged for all $f$ iff $\nu_{\rm sing} = 0$.
**Remark 2.** From the theorem it follows that $P_{\nu_1} = P_{\nu_2} \Rightarrow \nu_1 = \nu_2$.

**Proof.** W.l.o.g. we may assume that $\nu$ is positive. Let $f \in C_c(\mathbb{R})$ be given. Note that

$$\int_{\mathbb{R}} f(x)\mathrm{d}\nu_y(x) = \pi^{-1}\int_{\mathbb{R}} P_{(f(x)-f)\nu}(x+\mathrm{i}y)\mathrm{d}x + \int_{\mathbb{R}} f(t)\mathrm{d}\nu(t).$$

Hence, to prove the statement, it suffices to show that

$$\lim_{y\downarrow 0}\int_{\mathbb{R}} P_{|f(x)-f|\nu}(x+\mathrm{i}y)\mathrm{d}x = 0.$$

Since $f$ has a compact support, we may assume that $\nu(\mathbb{R}) < \infty$ (justify this). Fix $\epsilon > 0$. Since $f$ is uniformly continuous, $\exists\delta > 0$ such that $|x-t| < \delta \Rightarrow |f(x)-f(t)| < \epsilon$. Let $M = \sup|f(x)|$ and set

$$C(x) = 2M\int_{|x-t|\geq\delta}\frac{\mathrm{d}\nu(t)}{(x-t)^2}.$$

Then

$$\begin{aligned}\int_{\mathbb{R}} P_{|f(x)-f)|\nu}(x+\mathrm{i}y)\mathrm{d}x &\leq \pi\nu(\mathbb{R})\epsilon + y\int_{\mathbb{R}} C(x)\mathrm{d}x\\ &\leq \pi\nu(\mathbb{R})(\epsilon + 4yM/\delta).\end{aligned}$$

Hence,

$$\limsup_{y\downarrow 0}\int_{\mathbb{R}} P_{|f(x)-f|\nu}(x+\mathrm{i}y)\mathrm{d}x \leq \epsilon,$$

and since $\epsilon > 0$ is arbitrary, the statement follows. □

### 1.5. Local $L^p$-norms, $p > 1$

In this subsection we will prove Theorem 2.1 of [Si1].

Let $\nu$ be a complex or positive measure and let $\nu = fm + \nu_{\text{sing}}$ be its Radon-Nikodym decomposition w.r.t. the Lebesgue measure.

**Theorem 4.** Let $A \subset \mathbb{R}$ be open, $p > 1$, and assume that

$$\sup_{0<y<1}\int_A |P_\nu(x+\mathrm{i}y)|^p\mathrm{d}x < \infty.$$

Then:
(1) $\nu_{\text{sing}} \restriction A = 0$.
(2) $\int_A |f(x)|^p\mathrm{d}x < \infty$.
(3) For any $[a,b] \subset A$, $\pi^{-1}P_\nu(x+\mathrm{i}y) \to f(x)$ in $L^p([a,b],\mathrm{d}x)$ as $y \downarrow 0$.

**Proof.** We will prove (1) and (2), (3) is a homework problem.

Let $g$ be a continuous function with compact support contained in $A$ and let $q$ be the index dual to $p$, $p^{-1}+q^{-1}=1$. Then, by Theorem 3,

$$\int_A g\mathrm{d}\nu = \lim_{y\downarrow 0}\pi^{-1}\int_A g(x)P_\nu(x+\mathrm{i}y)\mathrm{d}x,$$

and

$$\left|\int_A g(x)P_\nu(x+\mathrm{i}y)\mathrm{d}x\right| \le \left(\int_A |g(x)|^q\mathrm{d}x\right)^{1/q}\left(\int_A |P_\nu(x+\mathrm{i}y)|^p\mathrm{d}x\right)^{1/p}$$

$$\le C\left(\int_A |g(x)|^q\mathrm{d}x\right)^{1/q}.$$

Hence, the map $g\to\int_A g(x)\mathrm{d}\nu(x)$ is a continuous linear functional on $L^q(A,\mathrm{d}x)$, and there is a function $\tilde{f}\in L^p(A,\mathrm{d}\mu)$ such that

$$\int_A g(x)\mathrm{d}\nu(x) = \int_A g(x)\tilde{f}(x)\mathrm{d}x.$$

This relation implies that $\nu\restriction A$ is absolutely continuous w.r.t. the Lebesgue measure and that $f(x)=\tilde{f}(x)$ for Lebesgue a.e. $x$. (1) and (2) follow. □

Theorem 4 has a partial converse which we will discuss in the homework set.

### 1.6. Local version of the Wiener theorem

In this subsection we prove Theorem 2.2 of [Si1].

**Theorem 5.** Let $\nu$ be a signed measure and $A_t$ be the set of atoms of $\nu$. Then for any finite interval $[a,b]$,

$$\lim_{y\downarrow 0} y\int_a^b |P_\nu(x+\mathrm{i}y)|^2\mathrm{d}x = \frac{\pi}{2}\left(\frac{|\nu(\{a\})|^2}{2}+\frac{|\nu(\{b\})|^2}{2}+\sum_{x\in(a,b)\cap A_t}|\nu(\{x\})|^2\right).$$

**Proof.**

$$|P_\nu(x+\mathrm{i}y)|^2 = y^2\int_{\mathbb{R}^2}\frac{\mathrm{d}\nu(t)\mathrm{d}\nu(t')}{((x-t)^2+y^2)((x-t')^2+y^2)}.$$

and

$$y\int_a^b |P_\nu(x+\mathrm{i}y)|^2\mathrm{d}x = \int_{\mathbb{R}^2} g_y(t,t')\mathrm{d}\nu(t)\mathrm{d}\nu(t'),$$

where

$$g_y(t,t') = \int_a^b \frac{y^3 \mathrm{d}x}{((x-t)^2+y^2)((x-t')^2+y^2)}.$$

Notice now that:
(1) $0 \le g_y(t,t') \le \pi$.
(2) $\lim_{y\downarrow 0} g_y(t,t') = 0$ if $t \ne t'$, or $t \notin [a,b]$, or $t' \notin [a,b]$.
(3) If $t = t' \in (a,b)$, then

$$\lim_{y\downarrow 0} g_y(t,t) = \lim_{y\downarrow 0} y^3 \int_{\mathbb{R}} \frac{\mathrm{d}x}{(x^2+y^2)^2} = \frac{\pi}{2}$$

(compute the integral using the residue calculus).
(4) If $t = t' = a$ or $t = t' = b$, then

$$\lim_{y\downarrow 0} g_y(t,t) = \lim_{y\downarrow 0} y^3 \int_0^{\infty} \frac{\mathrm{d}x}{(x^2+y^2)^2} = \frac{\pi}{4}$$

The result follows from these observations and the dominated convergence theorem, in the same way as in the proof of the Wiener theorem in the Lecture 1. □

**Corollary 1.** The signed measure $\nu$ has no atoms in $[a,b]$ iff

$$\lim_{y\downarrow 0} y \int_a^b |P_\nu(x+\mathrm{i}y)|^2 \mathrm{d}x = 0.$$

### 1.7. Poisson representation of harmonic functions

**Theorem 6.** Let $V(z)$ be a positive harmonic function in $\mathbb{C}_+$ ($V(z) \ge 0$). Then there is a constant $c \ge 0$ and a positive measure $\mu$ on $\mathbb{R}$ such that

$$V(x+\mathrm{i}y) = cy + P_\mu(x+\mathrm{i}y).$$

The $c$ and $\mu$ are uniquely determined by $V$.

**Remark 1.** The constant $c$ is unique since $c = \lim_{y\to\infty} V(\mathrm{i}y)/y$. This fact and Remark 2 after Theorem 3 yield that $\mu$ is also unique.
**Remark 2.** Theorems 1 and 6 yield that if $V$ is a positive harmonic function in $\mathbb{C}_+$ and $\mathrm{d}\mu = f(x)\mathrm{d}x + \mu_{\mathrm{sing}}$ is the associated measure, then for Lebesgue a.e. $x$

$$\lim_{y\downarrow 0} \pi^{-1} V(x+\mathrm{i}y) = f(x).$$

Let $D = \{z : |z| < 1\}$ and $\Gamma = \{z : |z| = 1\}$. For $z \in D$ and $w \in \Gamma$ let

$$p_z(w) = \mathrm{Re}\,\frac{w+z}{w-z} = \frac{1-|z|^2}{|w-z|^2}.$$

We will first prove:

**Theorem 7.** Let $U$ be a positive harmonic function in $D$. Then there exists a finite positive Borel measure $\nu$ on $\Gamma$ such that for all $z \in D$,

$$U(z) = \int_\Gamma p_z(w)\mathrm{d}\nu(w).$$

**Proof.** By the mean value property of harmonic functions, for any $0 < r < 1$,

$$U(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} U(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta.$$

In particular,

$$\sup_{0<r<1} \frac{1}{2\pi}\int_{-\pi}^{\pi} U(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta = U(0) < \infty.$$

For $f \in C(\Gamma)$ set

$$\Phi_r(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} U(re^{\mathrm{i}\theta})f(\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta.$$

Obviosuly, $\Phi_r$ are continuous linear functionals on the Banach space $C(\Gamma)$ and $\|\Phi_r\| = U(0)$. By the standard diagonal argument yields there is a sequence $r_j \to 1$ and a bounded linear functional $\Phi$ on $C(\Gamma)$ such that $\Phi_{r_j} \to \Phi$ weakly, that is, for all $f \in C(\Gamma)$, $\Phi_{r_j}(f) \to \Phi(f)$. Obviously, $\|\Phi\| = U(0)$. By the Riesz representation theorem there exists a complex measure $\nu$ on $\Gamma$, $|\nu|(\Gamma) = U(0)$, such that

$$\Phi(f) = \int_\Gamma f(w)\mathrm{d}\nu(w).$$

Since $\Phi_{r_j}(f) \geq 0$ if $f \geq 0$, the measure $\nu$ is positive. Finally, let $z \in D$. If $r_j > |z|$, then

$$U(zr_j) = \frac{1}{2\pi}\int_{-\pi}^{\pi} U(r_j\mathrm{e}^{\mathrm{i}\theta})p_z(\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta = \Phi_{r_j}(p_z). \tag{12}$$

(the proof of this relation is left for homework—see Theorem 11.8 in [Ru]). Taking $j \to \infty$ we derive

$$U(z) = \Phi(p_z) = \int_\Gamma p_z(w)\mathrm{d}\nu(w).$$

$\square$

Before proving Theorem 6, I would like to make a remark about the change of variable in measure theory. Let $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ be measure spaces. A map $T : M_1 \to M_2$ is called a measurable tranformation if for all $F \in \mathcal{F}_2$, $T^{-1}(F) \in \mathcal{F}_1$. Let $\mu$ be a positive measure on $(M_1, \mathcal{F}_1)$, and let $\mu_T$ be a positive measure on $(M_2, \mathcal{F}_2)$ defined by $\mu_T(F) = \mu(T^{-1}(F))$. If $f$ is a measurable function on $(M_2, \mathcal{F}_2)$, then $f_T := f \circ T$ is a measurable function on $(M_1, \mathcal{F}_1)$. Then $f \in L^1(M_2, \mathrm{d}\mu_T)$ iff $f_T \in L^1(M_1, \mathrm{d}\mu)$, and in this case

$$\int_{M_2} f \mathrm{d}\mu_T = \int_{M_1} f_T \mathrm{d}\mu.$$

This relation is easy to check if $f$ is a characteristic function. The general case follows by the usual approximation argument through simple functions.

If $T$ is a bijection, then $g \in L^1(M_1, \mathrm{d}\mu)$ iff $g_{T^{-1}} \in L^1(M_2, \mathrm{d}\mu_T)$, and in this case

$$\int_{M_1} g \mathrm{d}\mu = \int_{M_2} g_{T^{-1}} \mathrm{d}\mu_T.$$

**Proof of Theorem 6.** We define a map $S : \mathbb{C}_+ \to D$,

$$S(z) = \frac{\mathrm{i} - z}{\mathrm{i} + z}. \tag{13}$$

This is the well-known conformal map between the upper half-plane and the unit disc. The map $S$ extends to a homeomorphism $S : \overline{\mathbb{C}}_+ \to \overline{D} \backslash \{-1\}$. Note that $S(\mathbb{R}) = \Gamma \backslash \{-1\}$. If

$$K_z(t) = \frac{y}{(x-t)^2 + y^2}, \qquad z = x + \mathrm{i}y \in \mathbb{C}_+,$$

then

$$(1+t)^2 K_z(t) = p_{S(z)}(S(t)).$$

Let $T = S^{-1}$. Explicitely,

$$T(z) = \frac{z-1}{\mathrm{i}z + \mathrm{i}}. \tag{14}$$

Let $U(z) = V(T(z))$. Then there exists a positive finite Borel measure $\nu$ on $\Gamma$, such that

$$U(z) = \int_\Gamma p_z(w) \mathrm{d}\nu(w).$$

The map $T : \Gamma\backslash\{-1\} \to \mathbb{R}$ is a homeomorphism, and let $\nu_T$ be the induced Borel measure on $\mathbb{R}$. Then by the change of the variable,

$$\int_{\Gamma\backslash\{-1\}} p_z(w)\mathrm{d}\nu(w) = \int (1+t)^2 K_{T(z)}(t)\mathrm{d}\nu_T(t).$$

Hence, for $z \in \mathbb{C}_+$

$$V(z) = p_{S(z)}(-1)\nu(\{-1\}) + \int_{\mathbb{R}} (1+t)^2 K_z(t)\mathrm{d}\nu_T(t).$$

Since $p_{S(z)}(-1) = y$, setting $c = \nu(\{-1\})$ and $\mathrm{d}\mu(t) = (1+t^2)\mathrm{d}\nu_T(t)$, we derive the statement. □

**1.8. The Hardy class $H^\infty(\mathbb{C}_+)$**

The Hardy class $H^\infty(\mathbb{C}_+)$ is the vector space of all functions $V$ analytic in $\mathbb{C}_+$ such that

$$\|V\| = \sup_{z\in\mathbb{C}_+} |V(z)| < \infty. \tag{15}$$

$H^\infty(\mathbb{C}_+)$ with norm (15) is a Banach space. In this subsection we will prove two basic properties of $H^\infty(\mathbb{C}_+)$.

**Theorem 8.** Let $V \in H^\infty(\mathbb{C}_+)$. Then for Lebesgue a.e. $x \in \mathbb{R}$, the limit

$$V(x) := \lim_{y\downarrow 0} V(x+\mathrm{i}y), \tag{16}$$

exists. Obviously, $V \in L^\infty(\mathbb{R}, \mathrm{d}x)$.

**Theorem 9.** Let $V \in H^\infty(\mathbb{C}_+)$, $V(z) \not\equiv 0$, and let $V(x)$ be given by (16). Then

$$\int_{\mathbb{R}} \frac{|\log|V(x)||}{1+x^2}\mathrm{d}x < \infty.$$

In particular, for all $\alpha \in \mathbb{C}$, $V(z) \not\equiv \alpha$ iff the set $\{x \in \mathbb{R} : V(x) = \alpha\}$ has zero Lebesgue measure.

**Proof of Theorem 8.** Let $\mathrm{d}\nu_y(t) = V(t+\mathrm{i}y)\mathrm{d}t$. Then, for $f \in L^1(\mathbb{R}, \mathrm{d}t)$,

$$\left|\int_{\mathbb{R}} f(t)\mathrm{d}\nu_y(t)\right| \le \|V\| \int_{\mathbb{R}} |f(t)|\mathrm{d}t.$$

The map $\Phi_y(f) := \int_{\mathbb{R}} f\mathrm{d}\nu_y$ is a linear functional on $L^1(\mathbb{R}, \mathrm{d}t)$ and $\|\Phi_y\| \le \|V\|$. By the Banach-Alaoglu theorem, there a bounded linear functional $\Phi$ and a sequence $y_n \downarrow 0$ such that for all $f \in L^1(\mathbb{R}, \mathrm{d}t)$,

$$\lim_{n\to\infty} \Phi_{y_n}(f) = \Phi(f).$$

Let $V \in L^\infty(\mathbb{R}, \mathrm{d}t)$ be such that $\Phi(f) = \int_{\mathbb{R}} V(t)f(t)\mathrm{d}t$.

Let $f(t) = \pi^{-1}y((x-t)^2 + y^2)^{-1}$. A simple residue calculation yields

$$\Phi_{y_n}(f) = \pi^{-1}y \int_{\mathbb{R}} \frac{V(t + \mathrm{i}y_n)\mathrm{d}t}{(x-t)^2 + y^2} = V(x + \mathrm{i}(y + y_n)).$$

Taking $n \to \infty$, we get

$$V(x + \mathrm{i}y) = \pi^{-1}y \int_{\mathbb{R}} \frac{V(t)\mathrm{d}t}{(x-t)^2 + y^2}, \tag{17}$$

and Theorem 1 of these lecture notes yields the statement. □

**Remark 1.** As Eugene has pointed out, Theorem 8 can be also proven using Theorem 6. The above argument has the advantage that it extends to any Hardy class $H^p(\mathbb{C}_+)$.

**Remark 2.** In the proof we have also established the Poisson representation of $V$ (the relation (17)).

**Proposition 1. (Jensen's formula)** Assume that $U(z)$ is analytic for $|z| < 1$ and that $U(0) \neq 0$. Let $r \in (0,1)$ and assume that $U$ has no zeros on the circle $|z| = r$. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the zeros of $U(z)$ in the region $|z| < r$ listed with multiplicites. Then

$$|U(0)| \prod_{j=1}^{n} \frac{r}{|\alpha_j|} = \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |U(r\mathrm{e}^{\mathrm{i}t})|\mathrm{d}t \right\}. \tag{18}$$

**Remark.** The Jensen formula holds even if $U$ has zeros on $|z| = r$, but then the proof is a bit more involved. We will only need the above elementary version.

**Proof.** Set

$$V(z) = U(z) \prod_{j=1}^{n} \frac{r^2 - \overline{\alpha}_n z}{r(\alpha_n - z)}.$$

Then for some $\epsilon > 0$ $V(z)$ has no zeros in the disk $|z| < r + \epsilon$ and the function $\log|V(z)|$ is harmonic in the same disk (see Theorem 13.12 in [Ru]). By the mean value theorem for harmonic functions,

$$\log |V(0)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |V(r\mathrm{e}^{\mathrm{i}\theta})|\mathrm{d}\theta.$$

The substitution yields the statement. □

**Proof of Theorem 9.** Setting $U(\mathrm{e}^{\mathrm{i}t}) = V(\tan(t/2))$, we have that

$$\int_{\mathbb{R}} \frac{|\log |V(x)||}{1 + x^2} \mathrm{d}x = \frac{1}{2} \int_{-\pi}^{\pi} |\log |U(\mathrm{e}^{\mathrm{i}t})||\mathrm{d}t. \tag{19}$$

Hence, it suffices to show that the integral on the r.h.s. is finite.

In the rest of the proof we will use the same notation as in the proof of Theorem 6. Recall that $S$ and $T$ are defined by (13) and (14). Let $U(z) = V(T(z))$. Then, $U$ is holomorphic in $D$ and $\sup_{z \in D} |U(z)| < \infty$. Moreover, the change of the variable and the formula (17) yield that

$$U(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta - t)} U(e^{it}) dt.$$

(Work out the details—the change of variable exercise is done in detail in [Ko], pages 106-107.)

The analog of Theorem 1 for the circle yields that for Lebesgue a.e. $\theta$

$$\lim_{r \to 1} U(re^{i\theta}) = U(e^{i\theta}). \tag{20}$$

The proof is outlined in the homework.

We will now make use of the Jensen formula If $U(0) = 0$, let $m$ be such $U_m(z) = z^{-m}U(z)$ satisfies $U_m(0) \neq 0$ (if $U(0) \neq 0$, then $m = 0$). Let $r_j \to 1$ be a sequence such that $U$ has no zeros on $|z| = r_j$. Set

$$J_{r_j} := \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |U_m(r_j e^{it})| dt = -m \log r_j + \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |U(r_j e^{it})| dt.$$

The Jensen formula (applied to $U_m$) yields that $J_{r_i} \leq J_{r_j}$ if $r_i \leq r_j$. Write $\log^+ x = \max(\log x, 0)$, $\log^- x = -\min(\log x, 0)$. Note that

$$\sup_t \log^+ |U(e^{it})| \leq \sup_t |U(e^{it})| < \infty.$$

Fatou's lemma, the dominated convergence and (20) yield that

$$J_{r_1} + \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^- |U(e^{it})| dt \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |U(e^{it})| dt < \infty.$$

Hence,

$$\int_{-\pi}^{\pi} |\log |U(e^{it})|| dt < \infty,$$

and the identity (19) yields the statement. □

### 1.10. The Borel tranform of measures

Recall that the Borel transform $F_\mu(z)$ is defined by (2).

**Theorem 10.** Let $\mu$ be a complex or positive measure. Then:
(1) For Lebesgue a.e. $x$ the limit

$$F_\mu(x) = \lim_{y \downarrow 0} F_\mu(x + \mathrm{i}y)$$

exists.
(2) If $\mu \neq 0$, then

$$\int_{\mathbb{R}} \frac{|\log |F_\mu(x)||}{1 + x^2} \mathrm{d}x < \infty. \tag{21}$$

(3) If $\mu \neq 0$, then for any complex number $\alpha$ the set

$$\{x \in \mathbb{R} : F_\mu(x) = \alpha\} \tag{22}$$

has zero Lebesgue measure.
**Remark.** In particular, the theorem implies that $F_{\mu_1} = F_{\mu_2} \Rightarrow \mu_1 = \mu_2$.
**Proof.** We first show that

$$F_\mu(z) = \frac{R(z)}{G(z)} \tag{23}$$

where $R, G \in H^\infty(\mathbb{C}_+)$ and $G$ has no zeros in $\mathbb{C}_+$. If $\mu$ is positive, set

$$G(z) = \frac{1}{\mathrm{i} + F_\mu(z)}.$$

Then, $G(z)$ is holomorphic in $\mathbb{C}_+$, $|G(z)| \leq 1$ (since $\mathrm{Im} F_\mu(z) \geq 0$), and

$$F_\mu(z) = \frac{1 - \mathrm{i}G(z)}{G(z)}.$$

If $\mu$ is a complex measure, first decompose $\mu = (\mu_1 - \mu_2) + \mathrm{i}(\mu_3 - \mu_4)$, where $\mu_i$'s are positive measures, then decompose

$$F_\mu(z) = (F_{\mu_1}(z) - F_{\mu_2}(z)) + \mathrm{i}(F_{\mu_3}(z) - F_{\mu_4}(z)).$$

and (23) follows from the corresponding result for positive measures.

Proof of (1): Since by Theorems 8 and 9 for Lebesgue a.e. $x$ the limits $R(x) := \lim_{y \downarrow 0} R(x + \mathrm{i}y)$ and $G(x) := \lim_{y \downarrow 0} G(x + \mathrm{i}y)$ exist and $G(x)$ is non-zero (the limits are obviously finite), we derive that for Lebesgue a.e. $x$,

$$F_\mu(x) = \lim_{y \downarrow 0} F_\mu(x + \mathrm{i}y) = \frac{R(x)}{G(x)}.$$

Proof of (2): $F_\mu(x)$ is zero on a set of positive measure iff $R(x)$ is, and if this is the case, $R(z) \equiv 0$ and then $F_\mu(z) \equiv 0$. Since the linear span of the set of functions $\{(x-z)^{-1} : z \in \mathbb{C}_+\}$ is dense in the Banach space $C_0(\mathbb{R})$ (prove this), $\mu = 0$. Hence, if $\mu \neq 0$, then $R(z) \not\equiv 0$. Obviously,

$$|\log|F_\mu(x)|| \leq |\log|R(x)|| + |\log|G(x)||,$$

and (21) follows from Theorem 9.

Proof of (3): The sets $\{x : F_\mu(x) = \alpha\}$ and $\{x : R(x) - \alpha G(x) = 0\}$ have the same Lebesgue measure. If the second set has positive Lebesgue measure, then by Theorem 9, $R(z) = \alpha G(z)$ for all $z \in \mathbb{C}_+$, and $F_\mu(z) \equiv \alpha$. Since $\lim_{y\to\infty} |F_\mu(x + \mathrm{i}y)| = 0$, $\alpha = 0$, and so $\mu = 0$. Hence, if the set $\{x : F_\mu(x) = \alpha\}$ has positive Lebesgue measure, then $\alpha = 0$ and $\mu = 0$.

The final result of this section is the celebrated theorem of Poltoratskii [Po]:

**Theorem 11.** Let $\nu$ be a complex and $\mu$ a positive measure. Let $\nu = f\mu + \nu_{\rm s}$ be the Radon-Nikodym decomposition. Let $\mu_{\rm sing}$ be the part of $\mu$ singular with respect to the Lebesgue measure. Then

$$\lim_{y\downarrow 0} \frac{F_\nu(x + \mathrm{i}y)}{F_\mu(x + \mathrm{i}y)} = f(x) \qquad \text{for } \mu_{\rm sing} - \textit{a.e. } x. \tag{24}$$

This theorem will play the central role in our study of the spectral structure of Anderson type Hamiltonians. Note that theorem follows easily if $\mu_{\rm sing} = \mu_{\rm pp}$. Also, it is easy to see that in (24) one cannot replace $\mu_{\rm sing}$ by $\mu$.

Poltoratskii's proof of Theorem 11 is somewhat complicated, partly since it is done in the framework of a theory that is also concerned with other questions. A relatively simple proof of Poltoratskii's theorem has been recently found in [JL]. This new proof is based on the spectral theorem for self-adjoint operators and rank one perturbation theory, and will be discussed in the Lecture 7.

### 1.11. Homework

[1] (This problem is a part of Lecture 1: I forgot to assign it). Let $\mu$ be a positive Borel measure on $\mathbb{R}$ finite on compact sets and $f \in L^1(\mathbb{R}, \mathrm{d}\mu)$. Prove that

$$\lim_{r\downarrow 0} \frac{1}{\mu(I(x,r))} \int_{I(x,r)} |f(t) - f(x)| \mathrm{d}\mu(t) = 0, \qquad \text{for } \mu - \textit{a.e. } x.$$

(Hint: You may follow the proof of Theorem 7.7 in [Ru]).

[2] (1) Prove Lemma 3.
(2) Assume that $\mu$ satifies (3). Prove that the set of $x$ for which (5) holds is $G_\delta$ (countable intersection of open sets) in $\operatorname{supp}\mu$.

[3] Prove Part (3) of Theorem 4.

[4] Theorem 4 has the following converse. Assume that (1) and (2) hold. Then for $[a,b]\subset A$,

$$\sup_{0<y<1}\int_a^b |P_\nu(x+\mathrm{i}y)|^p \mathrm{d}x<\infty.$$

Prove this.

[5] The following extension of Theorem 5 holds: Let $\nu$ be a finite positive measure. Then for any $p>1$,

$$\lim_{y\downarrow 0} y^{p-1}\int_a^b |P_\nu(x+\mathrm{i}y)|^p \mathrm{d}x = C_p\left(\frac{\mu(\{a\})^p}{2}+\frac{\mu(\{b\})^p}{2}+\sum_{x\in(a,b)}\mu(\{x\})^p\right).$$

Prove this and compute $C_p$ in terms of gamma functions. (Hint: Read Remark 1 after Theorem 2.2 in [Si1]).

[6] Prove the formula (12).

[7] In this exercise we will study Poisson transform on the circle. Let $\Gamma=\{z : |z|=1\}$ and let $\mu$ be a complex measure on $\Gamma$. The Poisson transform of the measure $\mu$ is

$$P_\mu(z)=\int_\Gamma \frac{1-|z|^2}{|z-w|^2}\mathrm{d}\mu(w).$$

If we parametrize $\Gamma$ by $w=\mathrm{e}^{\mathrm{i}t}$, $t\in(-\pi,\pi]$ and denote the induced complex measure by $\mu(t)$, we have

$$P_\mu(r\mathrm{e}^{\mathrm{i}\theta})=\int_{-\pi}^{\pi}\frac{1-r^2}{1+r^2-2r\cos(\theta-t)}\mathrm{d}\mu(t).$$

Note also that if $\mathrm{d}\mu(t)=\mathrm{d}t$, then $P_\mu(z)=2\pi$. For $w\in\Gamma$ we denote $I(w,r)$ the arc of the length $2r$ centered at $w$. Let $\nu$ be a complex measure and $\mu$ a finite positive measure on $\Gamma$. The corresponding maximal function is defined by

$$M_{\nu,\mu}(w)=\sup_{r>0}\frac{|\nu|(I(w,r))}{\mu(I(w,r))}$$

if $x \in \operatorname{supp} \mu$, otherwise $M_{\nu,\mu}(w) = \infty$.
(1) Formulate Besicovitch covering lemma for the circle and convince yourself that it has the same proof as Theorem 6 in Lec 1. Then the analogs of Theorems 7 and Proposition 2 obviously hold for the circle.
(2) Prove the following bound: For all $r \in [0,1)$ and $\theta \in (-\pi, \pi]$,

$$\frac{|P_\nu(r\mathrm{e}^{\mathrm{i}\theta})|}{P_\mu(r\mathrm{e}^{\mathrm{i}\theta})} \leq M_{\nu,\mu}(\mathrm{e}^{\mathrm{i}\theta}).$$

You can either mimick the proof of Lemma 2, or follow the argument of the second part of the proof of Theorem 11.20 in [Ru].
(3) State and prove the analog of Theorem 1 for the circle.
(4) State and prove the analogs of Theorems 3 and 8 for the circle.

[8] In the part (4) of the previous exercise you have proven the relation (20). You may also prove this relation directly as follows: show that

$$\begin{aligned}\limsup_{r\to 1} |U(r\mathrm{e}^{\mathrm{i}\theta}) - U(\mathrm{e}^{\mathrm{i}\theta})| &\leq \limsup_{r\to 1} \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta - t)}|U(\mathrm{e}^{\mathrm{i}t}) - U(\mathrm{e}^{\mathrm{i}\theta})|\mathrm{d}t \\ &\leq \limsup_{\epsilon\downarrow 0} \frac{1}{2\epsilon}\int_{I(\theta,\epsilon)} |U(\mathrm{e}^{\mathrm{i}t}) - U(\mathrm{e}^{\mathrm{i}\theta})|\mathrm{d}t,\end{aligned}$$

and then use Problem 1.

[9] The goal of this problem is to reprove all the results of this section for the non-tangential limits. Our description of non-tangential limits follows [Po]. Let again $\Gamma = \{z : |z| = 1\}$ and $D = \{z : |z| < 1\}$. Let $w \in \Gamma$. We say that $z$ tends to $w$ non-tangentially, and write

$$\underset{\angle}{z \to w},$$

if $z$ tends to $w$ inside the region

$$\Delta_w^\varphi = \{z \in D \,:\, |\mathrm{Arg}(1 - z\overline{w})| < \varphi\}$$

for all $\varphi \in (0, \pi/2)$ (draw a picture). $\mathrm{Arg}(z)$ is the principal branch of the argument with values in $(-\pi, \pi]$. In the sector $\Delta_w^\varphi$ inscribe the circle centered at the origin (we denote it by $\Gamma_\varphi$). The two points on $\Gamma_\varphi \cap \{z : \mathrm{Arg}(1 - z\overline{w}) = \pm\varphi\}$ divide the circle into two arcs. The open region bounded by the shorter arc and the rays $\mathrm{Arg}(1 - z\overline{w}) = \pm\varphi$ is denoted $\mathrm{C}_w^\varphi$. Let $\nu$ and $\mu$ be as in problem 8.
(1) Let $\varphi \in (0, \pi/2)$ be given. Then there is a constant $C$ such that

$$\sup_{z\in \mathrm{C}_w^\varphi} \frac{|P_\nu(z)|}{P_\mu(z)} \leq C M_{\nu,\mu}(w) \qquad \text{for } \mu - a.e.\, w. \tag{25}$$

This is the key result which extends the radial estimate of [8] part (2). The passage from the radial estimate to (25) is similar to the proof of Harnack's lemma. Write the detailed proof following Lemma 1.2 of [Po].
(2) Once (1) is established, the rest or the argument follows as before. Let $\nu = f\mu + \nu_{\rm s}$ be the Radon-Nikodym decomposition. Prove that

$$\lim_{z \underset{\angle}{\to} w} \frac{P_\nu(z)}{P_\mu(z)} = f(w) \qquad \text{for } \mu - \ a.e.w.$$

If $\nu$ is a positive measure, prove that

$$\lim_{z \underset{\angle}{\to} w} \frac{P_\nu(z)}{P_\mu(z)} = \infty \qquad \text{for } \nu_{\rm s} - \ a.e. \, w.$$

(3) Rework parts (3) and (4) of [8] for non-tangential limits.
(4) Consider now $\mathbb{C}_+$. We say that $z$ tends to $x$ non-tangentially if for all $\varphi \in (0, \pi/2)$ $z$ tends to $x$ inside the cone $\{z : |\mathrm{Arg}(z-x) - \pi/2| < \varphi\}$. Let $T$ be the conformal mapping (14). Prove that $z \to w$ non-tangentially in $D$ iff $T(z) \to T(w)$ non-tangentially in $\mathbb{C}_+$. Using this observation extend all the results of this section to the non-tangential limits.

### 1.12. Project

Let $\nu$ be a complex or positive measure on $\mathbb{R}$ and $\mu$ a positive measure. Let

$$\mathrm{d}\nu_y = \frac{P_\nu(x+\mathrm{i}y)}{P_\mu(x+\mathrm{i}y)} \mathrm{d}\mu(x).$$

On the basis of Theorem 3 one may naively expect that $\nu_y \to \nu$ weakly as $y \downarrow 0$, but this may not be the case (take $\nu = \delta_0, \mu = \delta_1$). Can you find reasonable non-trivial sufficient conditions which ensure that $\nu_y \to \nu$ weakly? Can you find sufficient and necessary conditions?