MATH 743
SPECTRAL THEORY OF ANDERSON TYPE HAMILTONIANS
LECTURE 1

Preliminaries 1: The measure theory

O.1 References

[Ru] Rudin W.: *Real and Complex Analysis*, 3rd edition, McGraw-Hill, 1987.
[Fed] Federer H.: *Geometric Measure Theory*, Springer-Verlag, 1969.

1.1. Introduction

The prerequisites for this course are MATH 564 and 565, and I will assume that you are familiar with basic notions of measure theory and functional analysis. In particular, I will assume that you are familiar with Chapters 1-6 and 8 of [Ru]. In Subsections 1.2-1.5 for reference and notational purposes we will review some results of the measure theory. In Subsection 1.6 we will discuss briefly Fourier transforms of measures and prove the Wiener theorem. In Subsection 1.7 we will discuss differentiation of measures (the reference for this subsection is [Fed]). In Subsection 1.8 you will find your first homework.

1.2. Basic notions

Let M be a set and $\mathcal{F}$ a σ-algebra in M. The pair $(M, \mathcal{F})$ is called a measure space. We denote by χ_A the characteristic function of a subset $A \subset M$.

Let μ be a measure on $(M, \mathcal{F})$. We say that μ is concentrated on the set $A \in \mathcal{F}$ if $\mu(E) = \mu(E \cap A)$ for all $E \in \mathcal{F}$. If $\mu(M) = 1$, then μ is called a probability measure.

Assume that M is a metric space. The minimal σ-algebra in M that contains all open sets is called Borel σ-algebra. A measure on Borel σ-algebra is called a Borel measure. If μ is Borel, the complement of the largest open set $\mathcal{O}$ such that $\mu(\mathcal{O}) = 0$ is called the support of μ and is denoted by $\operatorname{supp} \mu$.

Assume that M is a localy compact metric space. A Borel measure μ is called *regular* if for every $E \in \mathcal{F}$,

$$\mu(E) = \inf\{\mu(V) \,:\, E \subset V,\, V \,\text{open}\}$$
$$= \sup\{\mu(K) \,:\, K \subset E,\, K \,\text{compact}\}.$$

Theorem 1. Let M be a localy compact metric space in which every open set is σ-compact (that is, a countably union of compact sets). Let μ be a Borel measure finite on compact sets. Then μ is regular.

The measure space we will most often encounter is $\mathbb{R}^n$ with the usual Borel σ-algebra. The measure space $\mathbb{R}^{\mathcal{N}}$, where $\mathcal{N}$ is a countable set, will also play an important role. Let us describe this measure space in some detail.

Let $\mathcal{N}$ be a countable set. Let $\mathcal{R}^{\mathcal{N}}$ be the vector space of all functions $\omega : \mathcal{N} \to \mathbb{R}$. Let $\{a_n\}_{n\in\mathcal{N}}$ be a sequence such that $a_n > 0$, $\sum_{n\in\mathcal{N}} a_n < \infty$. For $\omega_1, \omega_2 \in \mathcal{R}^{\mathcal{N}}$ set

$$d(\omega_1, \omega_2) = \sum_{n\in\mathcal{N}} a_n \frac{|\omega_1(n) - \omega_2(n)|}{1 + |\omega_1(n) - \omega_2(n)|}.$$

Then d is a metric and (M, d) is a complete, separable metric space (such space is sometimes called a *Polish space*). This metric space with the corresponding Borel σ-algebra will be the starting point in our definition of Anderson type Hamiltonians. Note that if $\mathcal{N}$ is finite and has n elements, then $\mathcal{R}^{\mathcal{N}} = \mathbb{R}^n$.

Exercise 1. A set $E \subset \mathcal{R}^{\mathcal{N}}$ is called a Borel cylinder if there exists $n_1, n_2, \cdots, n_j \in \mathcal{N}$ and Borel sets $B_{n_1}, \cdots, B_{n_j}$ in $\mathbb{R}$ such that

$$E = \{\omega \,:\, \omega(n_i) \in B_{n_i},\, 1 \le i \le j\}.$$

Prove that the Borel σ-algebra in (M, d) is the minimal σ-algebra that contains every Borel cylinder. In particular, the Borel σ-algebra in M is independent of the choice of the sequence a_n.

Note that $\mathbb{R}^{\mathcal{N}}$ is *not* locally compact if $\mathcal{N}$ is infinite. This is not a serious difficulty—in most cases it can be bypassed by considering $\dot{\mathbb{R}}^{\mathcal{N}}$, where $\dot{\mathbb{R}}$ is the one point compactification of $\mathbb{R}$ (i.e. circle).

1.3. Complex measures

Let $(M, \mathcal{F})$ be a measure space. Let $E \in \mathcal{F}$. A countable collection of sets $\{E_i\}$ in $\mathcal{F}$ is called a partition of E if $E_i \cap E_j = \emptyset$ for $i \neq j$ and $E = \cup_j E_j$. A *complex measure* on $(M, \mathcal{F})$ is a function $\mu : \mathcal{F} \to \mathbb{C}$ such that

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i), \tag{1}$$

for every $E \in \mathcal{F}$ and *every* partition $\{E_i\}$ of E. In particular, the series (1) is absolutely convegent.

Note that complex measures take only finite values. The usual positive measures, however, are allowed to take the value ∞. In the sequel, the term *positive measure* will refer to the standard definition of a measure on a σ-algebra which takes values in $\mathbb{R}^+ \cup \{\infty\}$.

The set function $|\mu|$ on $\mathcal{F}$ defined by

$$|\mu|(E) = \sup \sum_i |\mu(E_i)|,$$

where the supremum is taken over all partitions $\{E_i\}$ of E, is called the total variation of the measure μ.

Theorem 2. Let μ be a complex measure. Then:
(1) The total variation $|\mu|$ is a positive measure on $(M, \mathcal{F})$.
(2) $|\mu|(M) < \infty$.
(3) There exists a measurable function $h : M \to \mathbb{C}$ such that $|h(x)| = 1$ for *all* $x \in M$ and

$$\mu(E) = \int_E h(x) \mathrm{d}|\mu|(x),$$

for all $E \in \mathcal{F}$. The last relation is abreviated $\mathrm{d}\mu = h\mathrm{d}|\mu|$.

A complex measure μ is called regular if $|\mu|$ is a regular measure. Note that if ν is a positive measure, $f \in L^1(M, \mathrm{d}\nu)$ and

$$\mu(E) = \int_E f \mathrm{d}\nu,$$

then

$$|\mu|(E) = \int_E |f| \mathrm{d}\nu.$$

The integral of a complex measure is defined in the obvious way, $\int f \mathrm{d}\mu = \int f h \mathrm{d}|\mu|$.

Notation. (Important!) Let μ be a complex or positive measure and $f \in L^1(M, \mathrm{d}|\mu|)$. In this course, we will often denote by $f\mu$ the complex measure

$$(f\mu)(E) := \int_E f \mathrm{d}\mu.$$

Note that $|f\mu| = |f||\mu|$.

Every complex measure can be written as a linear combination of four finite positive measures. Let $h_1 = \mathrm{Re} h(x), h_2 = \mathrm{Im} h(x)$, $h_i^+ = \max(h_i(x), 0)$, $h_i^-(x) = -\min(h(x), 0)$, and $\mu_i^\pm = h_i^\pm |\mu|$. Then

$$\mu = (\mu_1^+ - \mu^-) + \mathrm{i}(\mu_2^+ - \mu_2^-).$$

A complex measure μ which takes values in $\mathbb{R}$ is called a *signed* measure. Such measure can be decomposed as

$$\mu = \mu^+ - \mu^-,$$

where $\mu^+ = (|\mu| + \mu)/2$, $\mu^- = (|\mu| - \mu)/2$. If $A = \{x \in M : h(x) = 1\}$, $B = \{x \in M : h(x) = -1\}$, then for $E \in \mathcal{F}$,

$$\mu^+(E) = \mu(E \cap A), \qquad \mu^-(E) = -\mu(E \cap B).$$

This fact is known as the Hahn decomposition theorem.

1.4. Riesz representation theorem

In this subsection we assume that M is a locally compact metric space.

A continuous function $f : M \to \mathbb{C}$ *vanishes at infinity* if $\forall \epsilon > 0$ there exists a compact set K_ϵ such that $|f(x)| < \epsilon$ for $x \notin K_\epsilon$. The vector space of all continuous functions that vanish at infinity is denoted $C_0(M)$. Note that

$$\|f\| = \sup_{x \in M} |f(x)|$$

is a norm on $C_0(M)$ and that $C_0(M)$ is a Banach space. We denote by $C_0(M)^*$ its dual. The following result is known as the Riesz representation theorem.

Theorem 3. Let $\phi \in C_0(M)^*$. Then there exists a unique regular complex Borel measure μ such that

$$\phi(f) = \int_M f \mathrm{d}\mu$$

for all $f \in C_0(M)$. Moreover, $\|\phi\| = |\mu|(M)$.

1.5. Radon-Nikodym theorem

Let $(M, \mathcal{F})$ be a measure space and ν_1, ν_2 complex measures concentrated on disjoint sets. We will call such measures mutually singular (or orthogonal) and write $\nu_1 \perp \nu_2$. If $\nu_1 \perp \nu_2$, then $|\nu_1| \perp |\nu_2|$.

Let ν be a complex measure and μ a positive measure. We say that ν is absolutely continuous w.r.t. μ, and write $\nu \ll \mu$, if $\mu(E) = 0 \Rightarrow \nu(E) = 0$. The following result is known as the Lebesgue-Radon-Nikodym theorem.

Theorem 3. Let ν be a complex measure and μ a positive σ-finite measure on $(M, \mathcal{F})$. Then there exists a unique pair of complex measures ν_{a} and ν_{s} such that $\nu_{\mathrm{a}} \perp \nu_{\mathrm{s}}$, $\nu_{\mathrm{a}} \ll \mu$, $\nu_{\mathrm{s}} \perp \mu$, and

$$\nu = \nu_{\mathrm{a}} + \nu_{\mathrm{s}}.$$

Moreover, there exists a unique $f \in L^1(\mathbb{R}, \mathrm{d}\mu)$ such that $\forall E \in \mathcal{F}$,

$$\nu_{\mathrm{a}}(E) = \int_E f \mathrm{d}\mu.$$

The Radon-Nikodym decomposition is abreviated as $\nu = f\mu + \nu_{\mathrm{s}}$ (or $\mathrm{d}\nu = f\mathrm{d}\mu + \mathrm{d}\nu_{\mathrm{s}}$).

If $M = \mathbb{R}^n$ and μ is the Lebesgue measure, we will use special symbols for the Radon-Nikodym decomposition. We will denote by ν_{ac} the part of ν which is a.c. w.r.t. the Lebesgue measure and by ν_{sing} the part which is singular with respect to the Lebesgue measure. A point $x \in \mathbb{R}^n$ is called an atom of ν if $\nu(\{x\}) \neq 0$. Let A_t be the set of all atoms. The set A_t is countable and $\sum_{x \in A_t} |\nu(\{x\})| < \infty$. The singular part ν_{sing} of ν can further decomposed as $\nu_{\text{sing}} = \nu_{\text{sc}} + \nu_{\text{pp}}$, where ν_{sc} is the singular continuous part and ν_{pp} is the pure point part. The pure point part is defined by

$$\nu_{\text{pp}}(E) = \sum_{x \in E \cap A_t} \nu(\{x\}).$$

1.6. Fourier transform of measures

Let μ be a complex Borel measure on $\mathbb{R}$. Its Fourier transform is defined by

$$\hat{\mu}(t) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}tx} \mathrm{d}\mu(x).$$

$\hat{\mu}(t)$ is also called the characteristic function of the measure μ. Note that

$$|\hat{\mu}(t+h) - \hat{\mu}(t)| \leq \int_{\mathbb{R}} \left| \mathrm{e}^{-\mathrm{i}hx} - 1 \right| \mathrm{d}|\mu|,$$

and so the function $\mathbb{R} \ni t \to \hat{\mu}(t) \in \mathbb{C}$ is uniformly continuous.

The following result is known as the Riemann-Lebesgue lemma.

Theorem 4. Assume that μ is absolutely continuous w.r.t. the Lebesgue measure. Then

$$\lim_{|t| \to \infty} |\hat{\mu}(t)| = 0. \tag{2}$$

The relation (2) may hold even if μ is singular w.r.t. the Lebesgue measure. The measures for which (2) holds are called Rajchman measures. A geometric characterization of such measures can be found in Lyons R.: Seventy years of Rajchman measures. J. Fourier Anal. Appl. Publ. Res., Kahane Special Issue, 363, (1995).

Recall that A_t denotes the set of atoms of μ. In this subsection we will prove the Wiener theorem:

Theorem 5. Let μ be a signed Borel measure. Then

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T |\hat{\mu}(t)|^2 \mathrm{d}t = \sum_{x \in A_t} |\mu(\{x\})|^2.$$

Proof: Note first that

$$|\hat{\mu}(t)|^2 = \hat{\mu}(t)\overline{\hat{\mu}(t)} = \int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}t(x-y)} \mathrm{d}\mu(x) \mathrm{d}\mu(y).$$

Let

$$K_T(x,y) := \frac{1}{T} \int_0^T \mathrm{e}^{-\mathrm{i}t(x-y)} \mathrm{d}t = \begin{cases} (1 - \mathrm{e}^{-\mathrm{i}T(x-y)})/(\mathrm{i}T(x-y)) & \text{if } x \neq y, \\ 1 & \text{if } x = y. \end{cases}$$

Then

$$\frac{1}{T}\int_0^T |\hat{\mu}(t)|^2 \mathrm{d}t = \int_{\mathbb{R}^2} K_T(x,y)\mathrm{d}\mu(x)\mathrm{d}\mu(y).$$

Obviously,

$$\lim_{T\to\infty} K_T(x,y) = \begin{cases} 0 & \text{if } x \neq y, \\ 1 & \text{if } x = y. \end{cases}$$

Since $|K_T(x,y)| \leq 1$, by the dominated convergence theorem we have that for all x,

$$\lim_{T\to\infty}\int_{\mathbb{R}} K_T(x,y)\mathrm{d}\mu(y) = \mu\{(x)\}.$$

By Fubini's theorem,

$$\int_{\mathbb{R}^2} K_T(x,y)\mathrm{d}\mu(x)\mathrm{d}\mu(y) = \int_{\mathbb{R}}\left[\int_{\mathbb{R}} K_T(x,y)\mathrm{d}\mu(y)\right]\mathrm{d}\mu(x),$$

and by the dominated convergence theorem,

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\int_0^T |\hat{\mu}(t)|^2\mathrm{d}t &= \int_{\mathbb{R}} \mu(\{x\})\mathrm{d}\mu(x) \\ &= \sum_{x\in A_t} |\mu(\{x\})|^2.\end{aligned}$$

1.7. Differentiation of measures

We will discuss the differentiation of Borel measures on $\mathbb{R}$. All the results of this subsection hold for Borel measures on $\mathbb{R}^n$ (a part of your homework will deal with this extension).

We start by collecting some preliminary results. The first theorem we will need is the Besicovitch covering lemma.

Theorem 6. Let A be a bounded set in $\mathbb{R}$ and, for each $x \in A$, let I_x be an open interval with center at x.
(1) There is a countable subcollection $\{I_j\}$ of $\{I_x\}_{x\in A}$ such that $A \subset \cup I_j$ and that each point in $\mathbb{R}$ belongs to at most two intervals in $\{I_j\}$, i.e. $\forall y \in \mathbb{R}$,

$$\sum_j \chi_{I_j}(y) \leq 2.$$

(2) There is a countable subcollection $\{I_{i,j}\}$, $i = 1, 2$, of $\{I_x\}_{x\in A}$ such that $A \subset \cup I_{i,j}$ and $I_{i,j} \cap I_{i,k} = \emptyset$ if $j \neq k$.

In the sequel we will refer to $\{I_i\}$ and $\{I_{i,j}\}$ as the *Besicovitch subcollections.*

Proof. $|I|$ denotes the length of the interval I. We will use the shorthand

$$I(x,r) := (x-r, x+r).$$

Setting $r_x := |I_x|/2$, we have $I_x = I(x, r_x)$.

Let $d_1 = \sup\{r_x : x \in A\}$. Choose $I_1 = I(x_1, r_1)$ from the family $\{I_x\}_{x\in A}$ such that $r_1 > 3d_1/4$. Assume that $I_1, \cdots, I_{j-1}$ are chosen and that $A_j = A \setminus \cup_{i=1}^{j-1} I_i$ is non-empty. Let $d_j = \sup\{r_x : x \in A_j\}$. Then choose $I_j = I(x_j, r_j)$ from the family $\{I_x\}_{x\in A_j}$ such that $r_j > 3d_j/4$. In this way be obtain a countable (possibly finite) subcollection $I_j := I(x_j, r_j)$ of $\{I_x\}_{x\in A}$.

Suppose that $j > i$. Then $x_j \in A_i$ and

$$r_i \geq \frac{3}{4}\sup\{r_x : x \in A_i\} \geq \frac{3r_j}{4}. \tag{3}$$

This observation yields that the intervals $I(x_j, r_j/3)$ are disjoint. Indeed, if $j > i$, then $x_j \notin I(x_i, r_i)$, and (3) yields

$$|x_i - x_j| > r_i = \frac{r_i}{3} + \frac{2r_i}{3} > \frac{r_i}{3} + \frac{r_j}{3}.$$

Since A is a bounded set and $x_j \in A$, the disjointness of $I_j = I(x_j, r_j/3)$ implies that if the family $\{I_j\}$ is infinite, then

$$\lim_{j\to\infty} r_j = 0. \tag{4}$$

The relation (4) yields that $A \subset \cup_j I(x_j, r_j)$. Indeed, this is obvious if there are only finitely many I_j's. Assume that there are infinitely many I_j's and let $x \in A$. By (4), there is j such that $r_j < 3r_x/4$, and by the defintition of r_j, $x \in \cup_{i=1}^{j-1} I_i$.

Finally, notice that if three intervals in $\mathbb{R}$ have a common point, then one of the intervals is contained in the union of the other two. Hence, by droping superfluous intervals from the collection $\{I_j\}$, we derive that $A \subset \cup_j I_j$ and that each point in $\mathbb{R}$ belongs to no more then two intervals I_j. This proves (1).

To prove (2), we renumarate I_j's as follows. To I_1 associate the number 0. The intervals to right of I_1 enumarate in sucession by positive integers, and the intervals to the left by negative integers. (The "sucession" is well-defined, since no point belongs simulteneously to three intervals). The intervals associated to even integers are mutually disjoint, and so are the intervals associated to odd integers. Finally, the interval associated to $2n$ we denote $I_{1,n}$ and the interval associated to $2n+1$ we denote by $I_{2,n}$. This construction yields (2). □

Let μ be a positive Borel measure on $\mathbb{R}$ finite on compact sets and let ν be a complex measure. The corresponding maximal function is defined by

$$M_{\nu,\mu}(x) = \sup_{r>0} \frac{|\nu|(I(x,r))}{\mu(I(x,r))}, \qquad x \in \operatorname{supp}\mu.$$

If $x \notin \operatorname{supp}\mu$ we set $M_{\nu,\mu}(x) = \infty$. In the homework problem 1 you are asked to prove that the function $\mathbb{R} \ni x \to M_{\nu,\mu}(x) \in [0, \infty]$ is Borel measurable.
Theorem 7. For any $t > 0$,

$$\mu\{x : M_{\nu,\mu}(x) > t\} \leq \frac{2}{t}|\nu|(\mathbb{R}).$$

Proof. Let $[a, b]$ be a bounded interval. Then every point x in $[a, b] \cap \{x : M_{\nu,\mu}(x) > t\}$ is a center of an open interval I_x such that

$$|\nu|(I_x) \geq t\mu(I_x).$$

Let $I_{i,j}$ be the Besicovitch subcollection of $\{I_x\}$. Then,

$$[a, b] \cap \{x : M_{\nu,\mu}(x) > t\} \subset \cup I_{i,j},$$

and

$$\begin{aligned}
\mu\left([a, b] \cap \{x : M_{\nu,\mu}(x) > t\}\right) &\leq \sum_{i,j} \mu(I_{i,j}) \\
&\leq \frac{1}{t}\sum_{i,j} |\nu|(I_{i,j}) \\
&= \frac{1}{t}\sum_{i=1}^{2} |\nu|(\cup_j I_{i,j}) \\
&\leq \frac{2}{t}|\nu|(\mathbb{R}).
\end{aligned}$$

The statement follows by taking $a \to -\infty$ and $b \to +\infty$. □

In the homework problem 3 you are asked to prove:
Proposition 1. Let A be a bounded Borel set. Then for any $0 < p < 1$,

$$\int_A (M_{\nu,\mu}(x))^p \mathrm{d}\mu(x) < \infty.$$

We will also need:
Proposition 2. Let ν_j be a sequence of Borel complex measures such that $\lim_{j\to\infty} |\nu_j|(\mathbb{R}) = 0$. Then there is a subsequence ν_{j_k} such that

$$\lim_{k\to\infty} M_{\nu_{j_k},\mu}(x) = 0 \qquad \text{for } \mu - a.e.\, x.$$

Proof. By Theorem 7, for each $k = 1, 2, \cdots$ we can find j_k so that

$$\mu\left\{x : M_{\nu_{j_k},\mu}(x) > 2^{-k}\right\} \leq 2^{-k}.$$

Hence,

$$\sum_{k=1}^{\infty} \mu\left\{x : M_{\nu_{j_k},\mu}(x) > 2^{-k}\right\} < \infty,$$

and so for μ-a.e. x, there is k_x such that for $k > k_x$, $M_{\nu_{j_k},\mu}(x) \leq 2^{-k}$. This yields the statement. □

We are now ready to prove the main theorem of this subsection.

Theorem 8. Let ν be a complex Borel measure and μ a positive Borel measure finite on compact sets. Let $\nu = f\mu + \nu_{\mathrm{s}}$ be the Radon-Nikodym decomposition. Then:
(1)

$$\lim_{r\downarrow 0} \frac{\nu(I(x,r))}{\mu(I(x,r))} = f(x), \qquad \text{for } \mu - a.e.\ x.$$

In particular, $\nu \perp \mu$ iff

$$\lim_{r\downarrow 0} \frac{\nu(I(x,r))}{\mu(I(x,r))} = 0, \qquad \text{for } \mu - a.e.\ x.$$

(2) Let in addition ν be positive. Then

$$\lim_{r\downarrow 0} \frac{\nu(I(x,r))}{\mu(I(x,r))} = \infty, \qquad \text{for } \nu_{\mathrm{s}} - a.e.\ x.$$

Proof. (1) We will split the proof into two steps.
Step 1. Assume that $\nu \ll \mu$, namely that $\nu = f\mu$. Let g_n be a continuous function with compact support such that $\int_{\mathbb{R}} |f - g_n| \mathrm{d}\mu < 1/n$. Set $h_n = f - g_n$. Then, for $x \in \operatorname{supp}\mu$ and $r > 0$,

$$\left|\frac{f\mu(I(x,r))}{\mu(I(x,r))} - f(x)\right| \leq \frac{|h_n|\mu(I(x,r))}{\mu(I(x,r))} + \left|\frac{g_n\mu(I(x,r))}{\mu(I(x,r))} - g_n(x)\right| + |g_n(x) - f_n(x)|.$$

Since g_n is continuous, we obviously have

$$\lim_{r\downarrow 0} \left|\frac{g_n\mu(I(x,r))}{\mu(I(x,r))} - g_n(x)\right| = 0,$$

and so for all n and $x \in \operatorname{supp}\mu$,

$$\limsup_{r\downarrow 0} \left|\frac{f\mu(I(x,r))}{\mu(I(x,r))} - f(x)\right| \leq M_{h_n\mu,\mu}(x) + |g_n(x) - f(x)|.$$

Let n_j be a subsequence such that $g_{n_j} \to f(x)$ for μ-a.e. x. Since $\int |h_{n_j}| \mathrm{d}\mu \to 0$ as $j \to \infty$, Proposition 2 yields that there is a subsequence of n_j (which we denote by the same letter) such that $M_{h_{n_j}\mu,\mu}(x) \to 0$ for μ-a.e. x. Hence, for μ-a.e. x,

$$\limsup_{r\downarrow 0} \left| \frac{f\mu(I(x,r))}{\mu(I(x,r))} - f(x) \right| = 0,$$

and (1) holds if $\nu \ll \mu$.
Step 2. To finish the proof of (1), it suffices to show that if ν is a complex measure such that $\nu \perp \mu$, then

$$\lim_{r\downarrow 0} \frac{|\nu|(I(x,r))}{\mu(I(x,r))} = 0, \qquad \text{for } \mu - a.e.\, x. \tag{5}$$

Let S be a Borel set such that $\mu(S) = 0$ and $|\nu|(\mathbb{R} \setminus S) = 0$. Then

$$\frac{|\nu|(I(x,r))}{(\mu + |\nu|)(I(x,r))} = \frac{\chi_S(|\nu| + \mu)(I(x,r))}{(\mu + |\nu|)(I(x,r))}. \tag{6}$$

By Step 1,

$$\lim_{r\downarrow 0} \frac{\chi_S(|\nu| + \mu)(I(x,r))}{(|\nu| + \mu)(I(x,r))} = \chi_S(x), \qquad \text{for } |\nu| + \mu - \ a.e.\, x. \tag{7}$$

Since $\chi_S(x) = 0$ for μ-a.e. x, (6) and (7) yield (5).

(2) Since ν is positive, $\nu(I(x,r)) \geq \nu_{\rm s}(I(x,r))$, and we may assume that $\nu \perp \mu$. By (6) and (7),

$$\lim_{r\downarrow 0} \frac{\nu(I(x,r))}{(\nu + \mu)(I(x,r))} = 1, \qquad \text{for } \nu - \ a.e.\, x,$$

and so

$$\lim_{r\downarrow 0} \frac{\mu(I(x,r))}{\nu(I(x,r))} = 0, \qquad \text{for } \nu - \ a.e.\, x.$$

This yields part (2). □

We finish this subsection with several remarks. If μ is the Lebesgue measure, then the results of this section reduce to the standard differentiation results discussed, for example, in Chapter 7 of [Ru]. The arguments in [Ru] are based on the Vitali covering lemma which is specific to the Lebesgue measure. The proofs of this subsection are based on the Besicovitch covering lemma and they apply to an arbitrary positive measure μ. In fact, the proofs directly extend to $\mathbb{R}^n$ (one only needs to replace the intervals $I(x,r)$ with the balls $B(x,r)$ centered at x and of radius r) if one uses the following version of the Besicovitch covering lemma.

Theorem 9. Let A be a bounded set in $\mathbb{R}^n$ and, for each $x \in A$, let B_x be an open ball with center at x. Then there is an integer N, which depends only on n, such that:

(1) There is a countable subcollection $\{B_j\}$ of $\{B_x\}_{x\in A}$ such that $A \subset \cup B_j$ and each point in $\mathbb{R}^n$ belongs to at most N balls in $\{B_j\}$, i.e. $\forall y \in \mathbb{R}$,

$$\sum_j \chi_{B_j}(y) \le N.$$

(2) There is a countable subcollection $\{B_{i,j}\}$, $i = 1, \cdots, N$, of $\{B_x\}_{x\in A}$ such that $A \subset \cup B_{i,j}$ and $B_{i,j} \cap B_{i,k} = \emptyset$ if $j \neq k$.

Unfortunately, unlike the proof of the Vitali covering theorem, the proof of Theorem 9 is somewhat long and complicated.

1.8. Homework

[1] Prove that the maximal function $M_{\nu,\mu}(x)$ is Borel measurable.

[2] Let μ be a positive measure on $(M, \mathcal{F})$ and let f be a measurable function. Let

$$m_f(t) = \mu\{x \,:\, |f(x)| > t\}.$$

Prove that for $p \ge 1$,

$$\int_M |f(x)|^p \mathrm{d}\mu(x) = p \int_0^\infty t^{p-1} m_f(t) \mathrm{d}t.$$

(Hint: See Exercise 24 in M. Reed, B. Simon: Methods of Modern Mathematical Phsyis II. Fourier Analysis, Self-Adjointness. Academic Press 1975).

[3] Prove Proposition 1. (Hint: Use Exercise 2.)

[4] Prove the Riemann-Lebesgue lemma (Theorem 4).

[5] Let μ be a complex Borel measure on $\mathbb{R}$. Prove that $|\mu_{\mathrm{sing}}| = |\mu|_{\mathrm{sing}}$.

[6] Let μ be a positive measure on $(M, \mathcal{F})$. A sequence of measurable functions f_n converges in measure to zero if

$$\lim_{n\to\infty} \mu(\{x : |f_n(x)| > \epsilon\}) = 0$$

for all $\epsilon > 0$. The sequence f_n converges almost uniformly to zero if for all $\epsilon > 0$ there is a set $M_\epsilon \in \mathcal{F}$, such that $\mu(M_\epsilon) < \epsilon$ and f_n converges *uniformly* to zero on $M \setminus M_\epsilon$.

Prove that if f_n converges to zero in measure, then there is a subsequence f_{n_j} which convereges to zero almost uniformly.

[7] Prove Theorem 9. (This is a bit hard—you may want to consult the literature).

[8] State and prove the analog of Theorem 8 in $\mathbb{R}^n$. Use Theorem 9 and write all the details.