MATH 589

HOMEWORK 2

This homework consists of two parts

PART I

[1] Solve the problems 14.-1—14.-23 from Burrill's book.

Problems 14-1—14-8 are fairly easy (notice how nice is the problem 14-2; it is easy given the hint). In the problem 14-9 you are asked to fill in the steps of a proof of the Lindberg direction of the Lindberg-Feller theorem. Although the details are available in the notes, write an independent proof for yourself following the hint. The problems 14-15—14-16 are important (they would be hard without the hints). Problem 14-15 asserts that given an i.i.d. sequence X_n (mean zero, variance one) such that X_n has an integrable characteristic function, then S_n/s_n has a density f_n which converges uniformly to the Gaussian. This improves on the usual central limit theorem. Problem 14-16 exhibits an example of X_n with a density (and with non-integrable c.f.) such that f_n does not converge to the Gaussian (while of course $S_n/s_n \to \mathcal{N}(0,1)$). These are important facts.

In the problem 14-21 you are asked to show that a bounded random variable X is infinitely divisible iff $X = \text{const}$. This is an important thing to know! Let $|X| \leq M$ and assume $\mathbb{E}(X) = 0$. Write $X = X_{n1} + \cdots + X_{nn}$, where X_{nk} are i.i.d. Prove that $|X_{nk}| \leq M/n$, and deduce that $\text{Var}(X) = n\text{Var}(X_{n1}) \leq M^2/n$ for all n. Hence $X = 0$.

[2] Let X_n be independent random variables with finite variance and mean zero. Set $S_n = X_1 + \cdots + X_n$, $s_n = \sqrt{\text{Var}(S_n)}$. Prove:

(1) If $n^{-1-\delta}\sum_{k=1}^n \mathbb{E}(|X_k|^{1+\delta}) \to 0$ for a positive δ, then $S_n/n \to 0$ in probability.

(2) If $s_n^{-2-\delta}\sum_{k=1}^n \mathbb{E}(|X_k|^{2+\delta} \to 0$ for a positive δ, then $S_n/s_n \to \mathcal{N}(0,1)$.

Of course, these results follow from theorems proven in class. Try to give independent, self-contained proofs. (2) is the celebrated Liapounov's theorem which has been the turning point for the entire Central Limit problem.

[3] (Answer to Ricky's question). Let X_{nk} be a double sequence (I will follow the notation in the notes). Assume $\underline{\sigma} = 0$, $m_n \to m$, $\sigma_n^2 \to \sigma^2$, and that for some $\gamma \in \mathbb{R}$ and all $\epsilon > 0$,

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{|x-m_{nk}-\gamma|>\epsilon} (x - m_{nk})^2 \mathrm{d}\mu_{X_{nk}} = 0.$$

If $\gamma = 0$, then of course $S_n \to \mathcal{N}(m, \sigma^2)$. Show that if $\gamma \neq 0$, then $S_n \to \mathcal{P}(a, b, \lambda)$, where

$$a = m - \sigma^2/\gamma, \qquad b = \gamma, \qquad \lambda = \sigma^2/\gamma^2.$$

Hint: Show that $S_n \to X$ with Kolmogorov's pair (m, K), where $K(x) = 0$ for $x < \gamma$ and $K(x) = \sigma^2$ for $x \geq \gamma$.

[4] (1) Let X_n be a sequence of $\mathcal{N}(m_n, \sigma_n^2)$ random variables. If $X_n \to X$ weakly, prove that X is $\mathcal{N}(m, \sigma^2)$ and that $m_n \to m$, $\sigma_n^2 \to \sigma^2$. (The easiest way is to use the Lévy-Khintchine representation, consult the notes). Hence, the collection of all normal random variables is closed under weak limits.
(2) Let X be a $\mathcal{N}(m, \sigma^2)$ random variable. Construct X_n with Poisson $\mathcal{P}(a_n, b_n, \lambda_n)$ distribution such that $X_n \to X$ weakly.

[5] Let $\alpha \in \mathbb{R}$ and let X_n be independent random variables such that

$$P(X_n = \pm n^\alpha) = 1/2.$$

Set $s_n = \sqrt{\mathrm{Var}(S_n)}$. Show that S_n/s_n converges weakly. How the limiting distribution depends on α?

[6] Let X_{nk} be a double sequence of random variables with finite variance and mean zero. Assume that $\underline{\sigma} = 0$ and that $\sigma_n^2 = 1$ for all n. Prove

$$S_n \to \mathcal{N}(0,1) \iff \sum_{k=1}^{k_n} X_{nk}^2 \to 1.$$

(The right limit is in law/probability.)

[7] Let $\zeta(x + \imath t)$, $x > 1$, $t \in \mathbb{R}$, be the Riemann function defined by

$$\zeta(x + \imath t) = \sum_{n>0} n^{-x-\imath t} = \prod_p (1 - p^{-x-\imath t}),$$

where p varies over all primes. Prove that for fixed $x > 1$,

$$\varphi(t) = \zeta(x + \imath t)/\zeta(x),$$

is an infinitely divisible characteristic function.

[8] Think of a particle moving in $\mathbb{R}^3$ as follows. Start at $x = 0$ and pick a direction E_1 from the unit sphere $|x| = 1$ ($|x|$ denotes the length of $x \in \mathbb{R}^3$) according to the uniform distribution,

$$P(E_1 \in A) = (4\pi)^{-1} \times \text{area}(A),$$

and jump to the place $S_1 = E_1$ in time $n = 1$. Pick now an independent direction E_2 with the same distribution and jump to $S_2 = E_1 + E_2$ at time $n = 2$. Continue in the same way, arriving at $S_n = E_1 + \cdots + E_n$ at time n. Prove that for any Borel set $V \in \mathbb{R}^3$,

$$\lim_{n\to\infty} P(n^{-1/2} S_n \in V) = (3/2\pi)^{3/2} \int_V e^{-3|x|^2/2} \mathrm{d}^3 x.$$

(This observation is due to Lord Rayleigh).

Hint: If $t = (t_1, t_2, t_3)$, show first that

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}t\cdot E_n}) = (4\pi)^{-1} \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi} \mathrm{e}^{\mathrm{i}|t|\cos\phi} \sin\phi \mathrm{d}\phi = \frac{\sin|t|}{|t|}.$$

Using this, show that

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}n^{-1/2} S_n \cdot t}) = \left[\frac{\sin(n^{-1/2}|t|)}{n^{-1/2}|t|}\right]^n,$$

and that

$$\lim_{n\to\infty} \mathbb{E}(\mathrm{e}^{\mathrm{i}n^{-1/2} S_n \cdot t}) = \mathrm{e}^{-|t|^2/6}.$$

Derive the statement from this result.

[9] Consider the problem [8] in $\mathbb{R}^2$. Show that in this case

$$\lim_{n\to\infty} P(n^{-1/2} S_n \in V) = \pi^{-1} \int_V e^{-|x|^2} \mathrm{d}^3 x.$$

Can you solve the same problem in $\mathbb{R}^d$?

[10] Consider double sequences X_{nk} such that

$$P(X_{nk} = 1) = p_{nk}, \qquad P(X_{nk} = 0) = 1 - p_{nk}.$$

Assume that $\lim_{n\to\infty} \max_{1\le k\le k_n} p_{nk} = 0$.

(1) Show that if $\lambda_n = \sum_{j=1}^{k_n} p_{nj} \to \infty$, then $(S_n - \lambda_n)/\sqrt{\lambda_n} \to \mathcal{N}(0,1)$.

(2) What is the family of all possible limit laws for the sums S_n? Find conditions for the convergence to each specific law.

PART II

On the next few pages you will find a section from Varadhan's lecture notes dealing with Central Limit Theorem. Varadhan gives a compact and elegant presentation of some optimal results. This section is not an easy reading, but the result is well worth the effort. Try to understand this section.