MATH 589

HOMEWORK 1

This homework consists of three parts

PART I

[1] Solve the problems 12.-1 - 12.-56 from Burrill's book.

[2] Prove the Khintchine law of large numbers using characteristic functions. Compare it with the proof from the previous term (Hint: Consult problem 13.-4 in Burrill's book).

[3] Let X_n be i.i.d. random varaibles with a Cauchy distribution with density $(\pi(1+x^2))^{-1}$. Prove that the weak law of large numbers does not hold for the sequence X_n.

[4] Let X_n be i.i.d. random variables with density

$$f(x) = \frac{c}{(1+x^2)\ln(1+x^2)}$$

(c is the normalization constant). The random variables X_n do not have finite expectation. Using characteristic functions, prove that the weak law of large numbers holds for the sequence X_n.

[5] (This is a bit hard, so you may want to consult the literature). I have mentioned in class that in general it is not true that that moments determine the distribution. Find an example of two discrete random variables X and Y with different distributions and finite moments such that $\mathbb{E}(X^n) = \mathbb{E}(Y^n)$ for all n. Is it possible to find such an example if the random variables X, Y are continuous (i.e. have densities)? Can X be continuous and Y discrete?

[6] A family $\mathcal{F}$ of probability distribution functions on $\mathbb{R}$ is said to be *totally bounded* if, given any sequence F_n in $\mathcal{F}$, there is a subsequence that converges weakly to some limiting *probability distribution* function F.

Prove that a family $\mathcal{F}$ is totally bounded if and only if the following holds:
(1)

$$\lim_{a\to\infty} \sup_{F\in\mathcal{F}} \mu_F\left(|x| \geq a\right) = 0.$$

(2)

$$\lim_{h\to 0} \sup_{F\in\mathcal{F}} \sup_{|t|\le h} |1-\varphi_F(t)| = 0.$$

Compare this result with Helly's compactness theorem. What is the essential difference?

[7] (Riemann-Lebesgue lemma) Let F be a probability distribution function with density. Prove that

$$\lim_{|t|\to\infty} \varphi_F(t) = 0.$$

PART II

This part of the homework deals with theorems from Burrill's book I have discussed in class and whose proofs were left for the homework. You should carefully write down the proofs of Theorems 12-6A, 12-6B, 12-7A, 12-7B, 12-7D, and Example 12-4E. Make sure that you understand all the steps. Make sure that you understand the proof of Theorem 12-5A for general bounded distribution functions (in the class I have discussed only probability distribution functions and I have outlined how the proof extends to the general case).

PART III

This part of the homework deals with some additional material. On the next few pages you will find attached photocopies of a section from Varadhan's lecture notes and of a section from Dym-McKean "Fourier series and integrals".

The first section deals with law of Iterated Logarithm. (The law of Iterated Logarithm is also discussed in Burrill's book, Section 13-7). The second section deals with Spitzer's identity and Sparre-Anderson theorem. You should read these sections, understand them, and write down the detailed proofs clarifying all the steps.